MIN-MAX METHOD AND GROMOV'S WAIST INEQUALITY

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Abstract. In this paper, we survey topics related to Gromov's waist inequality including Morse Theory, Geometric Measure Theory and non-displaceable fibers of symplectic manifolds.

A Few Thoughts

We had a very fruitful and enjoyable experience in this REU program. We read and learned a lot during the summer, and the guidance of Prof. Burns provided us with new insights to look at the world of geometry and math. It's quite a shame that we weren't able to deliver any major results, but the knowledge we gained on how to do research during this REU could be very valuable for us when we go to graduate schools in the future.

Acknowledgement

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1. Introduction

Waist inequality[7] states that for any continuous map from S^n to $\mathbb{R}^k$, there exists a fiber whose all ϵ-neighborhoods is at least as large (volume) as that of the corresponding ϵ-neighborhood of the standard equator S^{n-k}. In this paper, we will first talk about Morse theory, Geometric Measure Theory and Min-max method which were used to first prove the waist inequality (under mild hypothesis on regularity and genericity) by Almgren. We will then discuss how the waist can be

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related to non-displaceable fibers and median in symplectic geometry by looking at the special case on S^2.

2. Morse Theory

2.1. Morse Theory on Smooth Manifolds.

The general idea of Morse theory is to study the (non-degenerate) critical points of smooth real valued functions on a manifold M, called Morse functions, to obtain information about the topology of M, with the following theorem:

Theorem 2.1. [10] *If f is a differentiable function on a manifold M with no degenerate critical points, and if each $M^a := \{x \in M : f(x) \leq a\}$ is compact, then M has the homotopy type of a CW-complex, with one cell of dimension λ for each critical point of index λ, where λ is the number of negative eigenvalues of the hessian of f at that point.*

Clearly, the index plays an important role in the theory. Morse's lemma states that:

Lemma 2.2 (Morse,[10])**.** *Let p be a non-degenerate critical point for $f \in C^\infty(M)$. Then there is a local coordinate system $(y^1, ..., y^n)$ in a neighborhood U of p such that the identity*

$$f = f(p) - (y^1)^2 - ... - (y^\lambda)^2 + (y^{\lambda+1})^2 + ... + (y^n)^2$$

holds throughout U, where λ is the index of f at p.

In the next section we will also discuss some properties of the index in infinite dimensional case. For now, let's look at an easy but interesting example:

Corollary 2.3. *(Reeb,*[10]*) If M is a compact manifold and f is a differentiable function on M with only two critical points, both of which are non-degenerate, then M is homeomorphic to a sphere.*

Proof. The two critical points must be the minimum and maximum points. Say that $f(p) = 0$ is the minimum and $f(q) = 1$ is the maximum. Since the indices are 0 and n, Morse's lemma implies that the sets $M^\epsilon = f^{-1}[0, \epsilon]$ and $f^{-1}[1 - \epsilon, 1]$ are closed $n-$cells for small ϵ. It is enough to show the next lemma, which implies that M^ϵ is homeomorphic to $M^{1-\epsilon}$, and thus M is obtained by attaching an $n-$cell to another $n-$cell, which is homeomorphic to S^n. ☐

Lemma 2.4. [10] *Let f be a smooth real valued function on a manifold M. Let $a < b$, and suppose that the set $f^{-1}[a, b]$ is compact and contains no critical points of f, then M^a is diffeomorphic to M^b.*

Proof. (sketch) Since there is no critical point in $f^{-1}[a,b]$, the deffeomorphism can be constructed by following the gradient flow of f. □

Corollary 2.3 contains an interesting idea to look at the Morse functions. We will discuss a little about Reeb graph in section 3.

2.2. **Calculus of variation; Morse Theory on the path space.**

The idea of Morse theory can be applied to many infinite dimensional problems. A first example would be the path space of a smooth manifold, with the energy function. We follow the treatment in [10]. The standard treatment involves piecewise smooth paths, but for simplicity, let's only consider smooth paths.

Definition 2.5. *Let p,q be two points of a smooth manifold M. A smooth path from p to q is a smooth map $\omega : [0,1] \to M$ such that $\omega(0) = p$ and $\omega(1) = q$. The path space is the set of all smooth paths from p to q in M, denoted by $\Omega(M;p,q)$, or briefly Ω.*

Definition 2.6. *Given a Riemannian metric, the energy of $\omega \in \Omega$ from a to b, where $0 \le a \le b \le 1$, is defined as*

$$E_a^b(\omega) = \int_a^b ||\frac{d\omega}{dt}||^2 dt.$$

In particular, we denote E for E_0^1.

To do Morse theory, well-defined notions of differentiation and the hessian are needed. This can be done through calculus of variation:

Definition 2.7. *A variation of ω is a function $\tilde{\alpha} : (-\epsilon, \epsilon) \to \Omega$ for some $\epsilon > 0$, such that*
(1) $\tilde{\alpha}(0) = \omega$;
(2) the map $\alpha : (-\epsilon,\epsilon) \times [0,1] \to M$ defined by $\alpha(u,t) = \tilde{\alpha}(u)(t)$ is smooth;
(3) $\tilde{\alpha}(u)$ fixes the endpoints, i.e. $\alpha(u,0) = p, \alpha(u,1) = q, \forall u \in (-\epsilon,\epsilon)$.

The variation $\tilde{\alpha}$ may be considered as a "smooth path" in Ω. The "tangent vector" and "tangent space" can be defined as:

Definition 2.8. *The "tangent vector" $\frac{d\tilde{\alpha}}{du}(0)$ is defined to be the vector field W along ω given by*

$$W_t = \frac{\partial \alpha}{\partial u}(0,t).$$

The "tangent space" of Ω at a path ω, denoted by $T\Omega_\omega$, is defined to be the vector space consisting of all smooth vector fields W along ω that vanish at boundary points, i.e. $W(0) = W(1) = 0$.

Now we can define differentiation. Let F be a real valued function on Ω. Given $W \in T\Omega_\omega$, choose a variation $\tilde{\alpha}$ with $\tilde{\alpha}(0) = \omega$, $\frac{d\tilde{\alpha}}{du}(0) = W$, then the "derivative" of F is defined by

$$F_*(W) = \frac{d(F(\tilde{\alpha}(u)))}{du}\bigg|_{u=0}.$$

Immediately (with some calculation), we have the first variation formula:

Theorem 2.9.

$$\begin{aligned} E_*(W) = \frac{dE(\tilde{\alpha}(u))}{du}\bigg|_{u=0} &= -2\int_0^1 < \frac{\partial \alpha}{\partial u}(0,t), \frac{D}{dt}\frac{d\omega}{dt} > dt \\ &= -2\int_0^1 < W_t, \frac{D}{dt}\frac{d\omega}{dt} > dt. \end{aligned}$$

Corollary 2.10. *The path ω is a critical point for the energy function E if and only if ω is a geodesic.*

Proof. This follows from the first variation formula and the arbitariness of the choice of the variation $\tilde{\alpha}$. □

Remark 2.11. *The conclusion is still true if ω is piecewise smooth.*

Before we continue to discuss the Hessian of the energy function, let's proceed in another direction for a while. Notice that geodesics are also the critical points of the length function (with fixed end points), and the generalization of this idea leads to the Plateau's problem: the existence of a minimal surface with a given boundary, where the minimal surface is the critical point of the volume function (of manifolds).

Let's look at 2-dimensional surface embedded in $\mathbb{R}^3$ as an example. We follow the treatment in [3]. Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrized surface. Choose a differentiable function $h : U \to \mathbb{R}$. The "normal variation" of f is defined by

$$\begin{aligned} \phi : U \times (-\epsilon, \epsilon) &\to \mathbb{R}^3 \\ (x,t) &\mapsto f(x) + th(x)N(x) \end{aligned}$$

where N is the smooth unit normal vector field to f. ϕ induces a family of surfaces

$$\begin{aligned} f^t : U &\to \mathbb{R}^3 \\ x &\mapsto \phi(x,t) \end{aligned}$$

Let $A(t)$ denote the area function, then by basic calculations

$$A'(0) = -\int_U 2hH\sqrt{det(g)}$$

where H is the mean curvature of f, defined by the average of the principle curvatures, i.e. eigenvalues of the second fundamental form, and g is the first fundamental form.

Proposition 2.12. $A'(0) = 0 \Leftrightarrow H \equiv 0$.

This leads to another definition of minimal surfaces: manifolds with zero mean curvature. However, the Plateau's problem becomes much subtler in higher dimension, and we will discuss it in the next section.

Now we are back to the discussion of Hessian[10]. Mimicking the hessian in finite dimension, we define the Hessian of the energy function as follows. Let γ be a geodesic. Given vector fields $W_1, W_2 \in T\Omega_\gamma$, choose a 2-parameter variation

$$\alpha : U \times [0,1] \to M$$

where U is a neighborhood of (0,0) in $\mathbb{R}^2$, so that

$$\alpha(0,0,t) = \gamma(t), \frac{\partial \alpha}{\partial u_1}(0,0,t) = W_1(t), \frac{\partial \alpha}{\partial u_2}(0,0,t) = W_2(t).$$

Then the Hessian is defined to be

$$E_{**}(W_1, W_2) = \left. \frac{\partial^2 E(\tilde{\alpha}(u_1,u_2))}{\partial u_1 \partial u_2} \right|_{(0,0)},$$

where $\tilde{\alpha}(u_1,u_2) \in \Omega$ denotes the path $\tilde{\alpha}(u_1,u_2)(t) = \alpha(u_1,u_2,t)$. Our final interest in this section would be the Morse's index theorem. First we review the definitions of Jacobi fields and conjugate points:

Definition 2.13. *A vector field along a geodesic γ is called a Jacobi field if it satisfies the equation*

$$\frac{D^2 J}{dt^2} + R(V,J)V = 0$$

where R is the curvature tensor.

Definition 2.14. *Let γ be a geodesic, $p = \gamma(a), q = \gamma(b)$. p, q are conjugate along γ if there exists a non-zero Jacobi field J along γ which vanishes for $t = a$ and $t = b$. The multiplicity of p and q as conjugate points is equal to the dimension of the vector space consisting of all such Jacobi fields.*

Also, we review the definition for "index":

Definition 2.15. *The index λ of the Hessian is defined to be the maximum dimension of $T\Omega_\gamma$ on which $E_{**} : T\Omega_\gamma \times T\Omega_\gamma \to \mathbb{R}$ is negative definite.*

Theorem 2.16. *(Morse) The index is equal to the number of points $\gamma(t)$, with $0 < t < 1$, such that $\gamma(t)$ is conjugate to $\gamma(0)$ along γ; each such conjugate point being counted with its multiplicity. This index is always finite.*

This theory is much deeper than what we have reached, but we would stop here and turn to another subject. For further reading, we refer the reader to [2] and [10].

2.3. **Geometric Measure Theory; Min-max Method and Minimal Varieties.**

As is mentioned in the previous section, the Plateau's problem becomes subtler in higher dimension. The reason is because the problem involves a limit process, but the limit of surfaces (manifolds) can be very crazy. This leads to the development of geometric measure theory, which generalizes the concept of manifolds through functionals, such that the limit process behaves nicely and thus solutions to Plateau's problem can be obtained. Let's look at some basic definitions:[12]

Definition 2.17. *Let $U \subset \mathbb{R}^m$ be an open set and $D^n(U)$ denote the set of smooth compactly supported $n-$forms on U, with the usual convex topology, and a norm given by*

$$|\omega| = \sup_{x \in U} \sqrt{\omega(x) \cdot \omega(x)}$$

where $\cdot$ is the inner product of $\wedge^n(\mathbb{R}^m)^*$. An $n-$current in U is defined to be a continuous linear functional on $D^n(U)$.

Each current T has a notion of support, which is defined to be the intersection of all closed sets C in $\mathbb{R}^m$ such that $T(\omega) = 0$ whenever $\omega = 0$ on C. On the other hand, there is a generalization of the volume function:

Definition 2.18. *The mass function is given by*

$$\mathbb{M}(T) = \sup_{|\omega| \leq 1, \omega \in D^n(U)} T(\omega).$$

The most intuitive but important example would be the current induced by a manifold. Any manifold M gives rise to a current T by $T(\omega) = \int_M \omega$. Stoke's theorem says that $\int_M d\eta = \int_{\partial M} \eta$, which implies that we can define the boundary operator as below:

Definition 2.19. *The boundary of an $n-$current T is defined by*

$$\partial T(\omega) = T(d\omega), \omega \in D^{n-1}(U).$$

However, current as merely a linear functional is not enough to deal with the regularity issue, so we need a more restrictive class of currents, which generalizes rectifiable sets:

Definition 2.20. *A subset $E \subset \mathbb{R}^m$ is called $n-$rectifiable if $\mathcal{H}^n(E) < \infty$ and it can be $\mathcal{H}^n-$almost everywhere covered by countably many C^1 (or smooth) $n-$dimensional surfaces, where $\mathcal{H}$ is the Hausdorff measure.*

Definition 2.21. *An $n-$dimensional current T is called integer multiplicity rectifiable if there exists:*
(1) an $n-$rectifiable set E
(2) an integer-valued function θ, called the multiplicity function, with $\int_E \theta d\mathcal{H}^n < \infty$
(3) a choice of $n-$vector τ for E satisfying $\tau(x) = v_1 \wedge ... \wedge v_n$ with $v_1, ..., v_n$ being orthonormal
such that

$$T(\omega) = \int_E < \omega(x), \tau(x) > \theta(x) d\mathcal{H}^n(x), \forall \omega \in D^n(U),$$

where $<,>$ means evaluation.
The symbol $\mathcal{R}_{n,K}$ denotes $n-$dimensional integer multiplicity rectifiable currents supported on a compact set K.

Notice that the support and the rectifiable set E that appears in the integral representation of currents above are two geometric ways to look at currents. In particular, support$\subset \overline{E}$, the closure of E.

Also, we need to define the topology of the space:

Definition 2.22. *For each $T \in \mathcal{R}_{n,K}$, the flat norm is defined to be*

$$F_K(T) = \inf\{\mathbb{M}(R) + \mathbb{M}(S) : R \in \mathcal{R}_{n,K}, S \in \mathcal{R}_{n+1,K} \text{ and } T = R + \partial S\}.$$

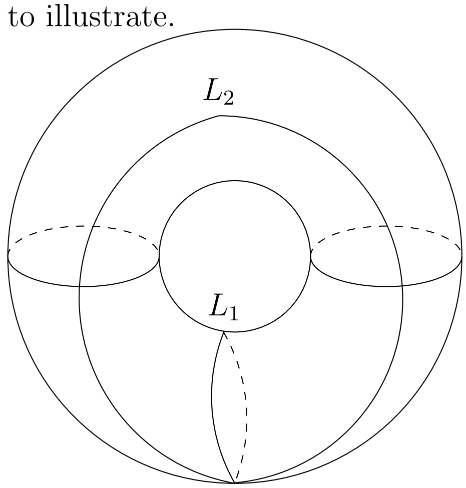
The reason for defining the norm as above is that we care about the limit process of both the interior and the boundary. Now we can state one of the fundamental theorems of geometric measure theory, the compactness theorem, which ensures a good behaviour of the limit process:

Theorem 2.23. [6] *Let K be a compact subset of $\mathbb{R}^m$ and $\{T_j\}$ a sequence in $\mathcal{R}_{n,K}$ such that*

$$\sup_j \{\mathbb{M}(T_j) + \mathbb{M}(\partial T_j)\} < \infty.$$

Then there exists $T \in \mathcal{R}_{n,K}$ with $\mathbb{M}(\partial T) < \infty$ and a subsequence T_{jl} that converges to T in flat topology.

Having set up the basic construction, we can proceed to discuss the application of Morse theory on the minimal surface problem. In section 2.1, we discussed how we can gain information about the topology of a manifold from the (non-degenerate) critical points of a smooth real valued function. The idea of the min-max method, on the contrary, is to tell the information about the critical points from the topological properties. Let's use the torus with a height function H as an example to illustrate.



T^2 torus has three homotopy types of loops (based on the bottom point). Ignoring the trivial type, there are two types L_1 and L_2 as is shown in the graph above. For each homotopy type L, we can define a "min-max" by

$$h = \inf_{l \in L} \sup_{x \in l} H(x).$$

Notice that there exists a point Q such that $H(Q) = h$, and Q is a critical point of the height function. An interesting way to describe this method is to "stretch" down the "strings" until it reaches the stable point, and at this moment the highest point reaches the "min-max" position.

In 1965, Almgren made progress on the minimal surface problem in the direction of Morse theory. He first computed all homotopy groups of the space of currents (more precisely, flat cycles), and the nontrivial topology suggests that the critical points for the area functional can be found by Morse theory. Then he applied the min-max method on the space, showing the existence of a minimal variety that reaches the "min-max" with respect to the mass function for each homotopy class[9]. Interestingly, in his paper, he also proved (a weaker version of) the waist inequality, and we will discuss it in the next section.

3. Waist Inequality and Non-displaceable Fibers

Theorem 3.1 (Gromov,[7]). *Let $f : S^n \longrightarrow \mathbb{R}^k$ be a continuous map where S^n is the unit n-sphere. Then there exist a point $z \in \mathbb{R}^k$ such that* *the spherical n-volumes of ϵ-neighborhoods of the level $Y_z = f^{-1}(z) \in S^n$, denoted $Y_z + \epsilon \subseteq S^n$, satisfy*

$$vol(Y_z + \epsilon) \geq vol(S^{n-k} + \epsilon)$$

for all $\epsilon > 0$, where $S^{n-k} \subseteq S^n$ denotes an equatorial $(n-k)$-sphere.

Observe that if $k = n$, Borsuk-Ulam theorem implies that $\exists x \in S^n$ such that $f(x) = f(-x)$ and the level set $Y_{f(x)}$ contains at least two antipodal points which is an equatorial 0-sphere. Therefore Gromov's waist inequality in the case of $k = n$ can be seen as a corollary of the Borsuk-Ulam theorem. On the other hand, if we assume $f^{-1}(z) \leq 2$ for all $z \in R^n$ and take $\epsilon = \frac{\pi}{2}$, Gromov's waist inequality implies there exists $x \in S^n$ such that $f(x) = f(-x)$ [7].

Theorem 3.2 (Levy's isoperimetric inequality for sphere)**.** *Let $\epsilon > 0$, A and B are two subset of S^n such that $vol(A) = vol(B)$. Further suppose that $B = \{x \in S^n | dist(x, a) \leq r\}$ where dist denotes distance on the sphere. Then $vol(A + \epsilon) \geq vol(B + \epsilon)$.*

Observe that if $k = 1$, then there exists $x \in \mathbb{R}$ such that $vol(f^{-1}(-\infty, x])$, $vol(f^{-1}[x, \infty)) \geq \frac{1}{2} vol(S^n)$, and $f^{-1}(x)$ is a waist of the function.

Definition 3.3. *A symplectic manifold is an even-dimensional manifold M^{2n} equipped with a closed 2-form ω that can be written as $\sum_{i=1}^n dp_i \wedge dq_i$ in some local coordinate (p, q).*

Definition 3.4 (Hofer,[8])**.** *For M a symplectic manifold, a subset $X \subseteq M$ is called displaceable if there exists a Hamiltonian diffeomorphism ϕ such that $\phi(X) \cap X = \emptyset$.*

Definition 3.5. *For M a symplectic manifold, Poisson bracket is a canonical operation on pair of smooth functions on M given by $\{f, g\} = \sum_i (\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i})$.*

Theorem 3.6 (Entov,[4])**.** *Let $\vec{f} = (f_1, ..., f_k) : M \longrightarrow \mathbb{R}^k$ be a smooth map on a closed symplectic manifold M whose coordinate functions f_i pairwise Poisson commute: $\{f_i, f_j\} = 0$. Then f has a non-displaceable fiber,i.e. $\exists x \in \mathbb{R}^k$ such that $f^{-1}(x)$ is non-empty and non-displaceable.*

The waist is a large fiber on S^n in the sense that its ϵ-neighborhood is at least as large as that of the equator while the non-displaceable fiber is a large fiber on a symplectic manifold M in the sense that it can't be moved away from itself. So a natural question would be to ask how the two concepts are related to one another and that's where we should look at the S^2 case since it is the only case where S^n is a symplectic manifold.

3.1. **n=2 case.** It is worth noticing that not all waists of smooth functions are non-displaceable. In fact, there is a convenient way to check if some smooth simple closed loop on S^2 is non-displaceable or not.

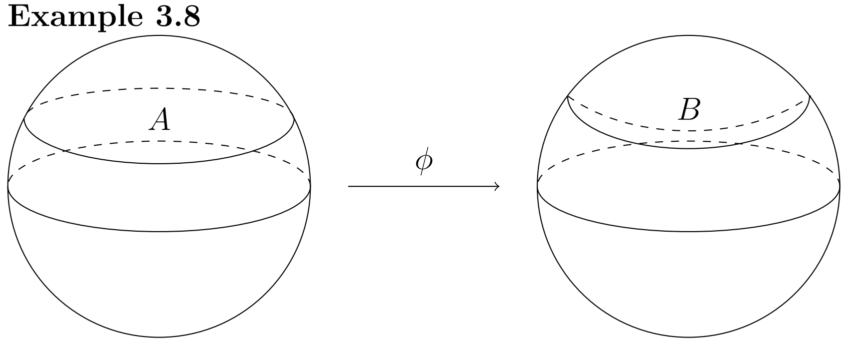
Theorem 3.7 (Moser,[11]). *Let M, N be two manifolds with volume element τ and σ, suppose that M, N can be mapped onto each other by a diffeomorphism $\phi : M \longrightarrow N$. If*

$$\int_M \tau = \int_N \sigma$$

then there exists a diffeomorphism of M onto N taking also τ to σ.

If we take M and N to be connected subset of S^2 whose areas are the same and whose boundaries are both smooth simple closed loops, then by theorem 3.7 we can find a symplectomorphism that maps M, N onto each other. Here's an example to help understand this theorem

Example 3.8

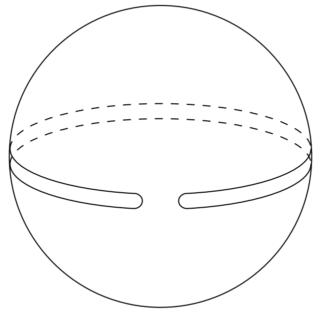


A is the "cap" on S^2, and B is the part above the curve on S^2 with the same area as A. Theorem 3.7 states that there exists a symplectomorphism that map A, B onto each other.

Using this, we can easily find examples of fibers being waist of some function but still displaceable.

Example 3.9

Let $S^2 = \{(x, y, z)|x^2 + y^2 + z^2 = 1\}$. Consider $f : S^2 \longrightarrow \mathbb{R}$ where $f(x, y, z) = x - 10^{15}y^{10}$ (we don't necessarily need the 10^{15} but we do want the constant to be large enough so that the preimage of some number close to 1 is close enough to the equator). If we look at the fiber $f^{-1}(0.99)$, the image on S^2 would be:



$vol(f^{-1}(0.99)+\epsilon)$ is approximately the same as $vol(g^{-1}(0.001)+\epsilon)$ for $g(x,y,z)=z^2$, and is much larger than the volume of ϵ-neighborhoods of the equator.

Theorem 3.10. *If a fiber is a smooth simple closed curve that is non-displaceable, then it is a waist.*

Proof. By theorem 3.7, such fiber divides the sphere into 2 parts with same area, denoted as A and B. By theorem 3.2, $vol(A+\epsilon)+vol(B+\epsilon) \geq vol(S^1+\epsilon)$, therefore the ϵ-neighborhood of the fiber is larger than that of the equator. □

Definition 3.11 (Reeb Graph)**.** *For a Morse function $f : S^2 \longrightarrow \mathbb{R}$, the Reeb Graph Γ of f is obtained from S^2 by collapsing all the connected component of the fibers of f to points. Denote $\pi : S^2 \longrightarrow \Gamma$ the natural projection onto the Reeb Graph. Denote μ the probability measure on Γ which is obtained by pushing forward the area on the sphere.*

Note that since S^2 is simply-connected, Γ in this case is in fact a tree.

Definition 3.12. *Let (Γ,μ) be a measured tree. A point $x \in \Gamma$ is called the median if the measure of each connected component of $\Gamma \setminus \{x\}$ is not greater than $\frac{1}{2}$*

Theorem 3.13 (Entov, Polterovich, [5])**.** *Every Reeb Graph Γ obtained from S^2 has a unique median.*

Theorem 3.14. *The preimage of the median x is non-displaceable.*

Proof. Proof by contradiction. Let A be the largest connected component in $S^2 \setminus \pi^{-1}(x)$. Since the measure of each connected component is not greater than $\frac{1}{2}$, $vol(A) \leq \frac{1}{2}vol(S^2)$. Suppose there exists a symplectomorphism ϕ such that $\phi(\pi^{-1}(x)) \cap \pi^{-1}(x) = \emptyset$, then $\phi(A) \cap A = \emptyset$. Since ϕ is area preserving, there exists some other connected component B such that $\phi(B) \cap A \neq \emptyset$ therefore $(\overline{\phi(B)} \setminus \phi(B)) \cap A \neq \emptyset$

which implies $\phi(\pi^{-1}(x)) \cap A \neq \emptyset$. Since $\overline{\phi(A)} \setminus \phi(A) \subseteq \phi(\pi^{-1}(x))$, and $\phi(\pi^{-1}(x))$ is connected, $\phi(\pi^{-1}(x)) \cap \pi^{-1}(x) \neq \emptyset$. Therefore $\pi^{-1}(x)$ is non-displaceable.

□

Intuitively, the longer and more "uniformly distributed" the fiber is, the more likely it is a waist. It seems natural that if the fiber is connected and partitioned the sphere into multiple parts then it's quite likely to have large ϵ-neighborhood.

Furthermore, even if we don't assume the connectedness condition, the non-displaceability would place restrictions on the area of each connected component after removing the fibers. Since the more components there are, the more likely the fibers having large neighborhood. We have the following conjecture:

Conjecture 3.15. *The non-displaceable fiber on S^2 is a waist.*

Besides fibers on S^2, another interesting topic is the relation between waist and non-displaceable fibers on the complex projective space $\mathbb{CP}^n$. It turns out that waist on $\mathbb{CP}^n$ can be defined in a similar manner.

Corollary 3.16. [1] *Let $f : \mathbb{CP}^n \longrightarrow \mathbb{C}^k$ be a continuous function, then there exists $x \in \mathbb{C}^k$ such that $vol(f^{-1}(x) + \epsilon) \geq vol(\mathbb{CP}^{n-k} + \epsilon)$ where $\mathbb{CP}^{n-k}$ is the standard complex projective $(n-k)$-subspace in $\mathbb{CP}^n$, and the volume form is induced by the Fubini-Study metric.*

Now that waist is defined on $\mathbb{CP}^n$, one possible future research direction is to find out if there's any relation among waist, non-displaceable fiber on $\mathbb{CP}^n$.

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