Symplectic Geometry Through Polytopes

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Abstract

We begin by exploring basic notions in symplectic geometry, leading up to a discussion of moment maps. These functions encapsulate quantities conserved by symmetries of a lie group acting on a symplectic manifold. This exploration culminates in Delzant's theorem which colloquially says that certain symplectic manifolds are classified by particular polytopes.

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1 Introduction

Below is an expository account of some results in symplectic and algebraic geometry; however, the symplectic view point is taken here. The main result is Delzant's theorem, which puts 'nice enough' symplectic manifolds equipped with a certain torus action in correspondence with 'nice enough' polytopes. Properties of the manifolds are able to be studied by examining the combinatorics of the polyhedra, which is often much more straight forward. Proofs of results are shirked in order to cover more material, but can be found in the references. Background and basic examples are provided so that the paper is accessible to undergraduates in math.

This write-up's content covers one of a few topics I explored during the summer of 2018 as part of the University of Michigan's Research Experience for Undergraduates program. Other topics ranged within differential, symplectic, and algebraic geometry. Please feel free to contact me for a more thorough account of my experience. I would like to thank Sarah Koch, Karen Smith, David Speyer, and Alaina Wollner for their parts in organizing the University of Michigan REU program and for affording me the opportunity to spend this summer learning math. I am especially grateful for my mentor's dedication and insights.

2 Symplectic Geometry Background

This section allows perusal of the symplectic geometry ideas necessary to understand moment maps and some related, powerful theorems.

Definition 1. A *symplectic manifold* (M, ω) is a pairing of a manifold M with a closed 2-form ω that is nowhere degenerate on M.

Remark 1. Nondegeneracy of ω implies that $\omega \wedge ... \wedge \omega = \omega^k$ for $2k = \dim(M)$ is a volume form for M. This gives that $\dim(M)$ must be even, and M must be orientable.

Example 1. Consider $(M, \omega) = (\mathbb{R}^{2n}, \sum_{j=1}^{n} dx_j \wedge dy_j)$ where $(x_1, ..., x_n, y_1, ..., y_n)$ are the coordinates for $\mathbb{R}^{2n}$. Note that with the identification $z_j = x_j + iy_j$, this symplectic form is equivalent to $\frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$ for the manifold $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

Example 2. (M, ω) be $(S^2,\ d\theta \wedge dh)$ where (θ, h) are cylindrical coordinates for $\mathbb{R}^3$ on the sphere away from the poles.

Definition 2. Suppose (M_1, ω_1) and (M_2, ω_2) are symplectic manifolds. A diffeomorphism $\varphi : M_1 \to M_2$ is called a *symplectomorphism* provided that $\varphi^*\omega_2 = \omega_1$. In that case, M_1 and M_2 are said to be *symplectomorphic*.

3 Moment Maps

Now we turn to studying cases in which our symplectic manifold is equipped with an action of a lie group that gives rise to a function called a *moment map*. A moment map encapsulates quantities conserved by symmetries of the lie group. This topic relates back to classical mechanics and the Noether principle. For the following definitions, let (M, ω) be a connected symplectic manifold of dimension $2n$.

First we introduce a notion that associates a smooth function on our manifold to a vector field on our manifold:

Definition 3. A *hamiltonian vector field* X on M is one such that the contraction $\iota_X \omega$ is exact. A smooth function $H : M \to \mathbb{R}$ is called a *hamiltonian function* for the hamiltonian vector field X provided that $\iota_X \omega = dH$.

Given a hamiltonian function H, since ω is nondegenerate, we can always calculate an associated hamiltonian vector field X_H. Conversely, a hamiltonian vector field X can (since $\iota_X \omega$ is exact) give rise to hamiltonian functions equivalent up to additive constants.

Next we associate a certain function to a group action on our manifold:

Definition 4. For a smooth action $\psi : G \times M \to M$, a *moment map*

$$\mu : M \to \mathfrak{g}^*$$

is a function with the following properties:

- The dual function $\mu^X : M \to \mathbb{R}$ defined by $\mu^X(p) := \langle \mu(p), X \rangle$ is a hamiltonian function for the vector field $X^\#$ defined by

$$X_p^\# = \frac{d}{dt}\Big|_{t=0} [\psi(\exp(-tX), p)].$$

  That is,

$$d\mu^X = \iota_{X^\#} \omega.$$

- The moment map μ is equivariant with respect to both ψ (the action of G) and the coadjoint action. That is, for all $g \in G$, $\mu \circ \psi_g = \mathrm{Ad}_g^* \circ \mu$, or, equivalently, that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\psi_g"] \arrow[d, "\mu"] & M \arrow[d, "\mu"] \\ \mathfrak{g}^* \arrow[r, "\mathrm{Ad}_g^*"] & \mathfrak{g}^* \end{tikzcd}$$

An action that gives rise to a moment map is called a *hamiltonian action*.

Remark 2. We will mainly concern ourselves with torus actions, in which case the coadjoint action is trivial; the second property for a moment map is reduced to G-equivariance: $\mu \circ \psi_g = \mu \ \forall g \in G$. Furthermore, $\mathfrak{g}^* \simeq \mathbb{R}^d$ for d the dimension of the torus, so we can simplify computations nicely by solving for μ one coordinate at a time by taking X_j to be the j^{th} standard basis vector of $\mathbb{R}^d$ and calculating $d\mu^{X_j} = \iota_{X_j^\#} \omega$.

Example 3. Consider the symplectic manifold $(S^2, d\theta \wedge dh)$ where (θ, h) are cylindrical coordinates as in example 2 above. Now let S^1 act on S^2 by rotations about the z axis; i.e. for $\rho \in S^1$, $(\theta, h) \in S^2$,

$$\psi_\rho(\theta, h) = (\theta + \rho, h).$$

Computing, we obtain that

$$X^\# = \pm \frac{\partial}{\partial \theta}$$

with sign depending on convention and direction of rotation. So we can solve:

$$\begin{aligned} d\mu^X &= \iota_{X^\#}\omega \\ &= \iota_{X^\#} d\theta \wedge dh - d\theta \wedge \iota_{X^\#} dh \\ &= dh. \end{aligned}$$

Hence, our moment map is given by the height function on the sphere.[1] Computations aside, this result for the moment map makes intuitive sense, as height is conserved by this lie group action. See figure 1 for a visual.

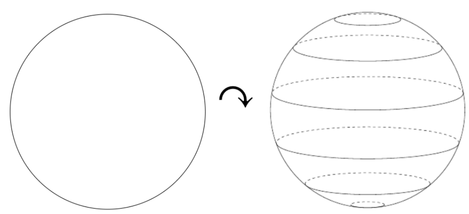

Figure 1: S^1 acting on S^2 by rotations about the z axis conserves height as shown by the lines of latitude which each depict the orbit of a point at that latitidue.

With a better understanding of moment maps, let us now turn our attention to their images and two powerful theorems: the convexity theorem and Delzant's theorem.

3.1 Convexity Theorem and Moment Polytopes

With a few conditions on our symplectic manifolds and group actions, the images of the moment maps are necessarily convex polytopes. The following theorem comes to us from Atiyah and Guillemin-Sternberg. Again, let (M, ω) be a symplectic manifold of dimension $2n$.

Theorem 1 (Convexity Theorem)**.** Suppose M is compact, connected, and equipped with a hamiltonian action ψ of the torus $\mathbb{T}^d$ with associated moment map

$$\mu : M \to \mathfrak{t}^* \simeq \mathbb{R}^d.$$

Let $\text{Fix}(\psi) = \{p \in M : \psi_g(p) = p \ \forall g \in G\}$. Under these assumptions, $\mu(\text{Fix}(\psi))$ is a finite set, and $\mu(M)$ is the convex hull of $\mu(\text{Fix}(\psi))$.

[1] See appendix A.1 for some discussion on the technicalities of this final portion of computation.

Definition 5. If the conditions of theorem 1 hold, $\mu(M)$ is called a *moment polytope.*

Let us return to a familiar example to illustrate this theorem.

Example 4. Recall that S^1 acting on $(S^2, d\theta \wedge dh)$ by rotating around the z axis has moment map given by the height function on the sphere. The fixed points of this action are the south and north poles; symbolically, $\mathrm{Fix}(\psi) = \{(0, -1),\ (0, 1)\}$. The image of this set under the moment map is finite: $\mu(\mathrm{Fix}(\psi)) = \{-1, 1\}$. Moreover, the moment polytope $\mu(M)$ is the interval $[-1, 1]$. See figure 2 for reference.

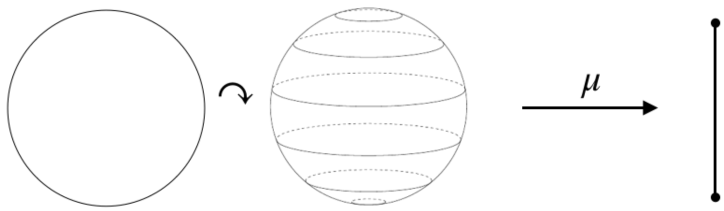

Figure 2: S^1 acting on S^2 by rotations about the z axis demonstrates theorem 1.

3.2 Delzant's Theorem

By adding two more stipulations to the hypotheses of the convexity theorem, we are able to say a lot about the properties of the moment polytopes. Furthermore, given a polytope with these certain properties, a symplectic manifold and group action can always be associated to it.

First we will provide terminology that encapsulates the required conditions for our manifolds and for our polytopes, then we will present the theorem that relates them.

Definition 6. A *symplectic toric manifold* of dimension $2n$ is a compact, connected, symplectic manifold equipped with an effective hamiltonian action by a torus of dimension n. See example 3.

Definition 7. An n-dimensional convex polytope is called *Delzant* provided that it has the following properties:

- simplicity: there are n edges meeting at each vertex p,
- rationality: the edges meeting at p are of the form $p + tv_i$, for some $t > 0$ and $v_i \in (\mathbb{Z}^n)^*$,
- smoothness: the set of v_i from above can be chosen as a $\mathbb{Z}$ basis of $(\mathbb{Z}^n)^*$.

Example 5. See figure 3 for polytopes that fail to be Delzant, and see figure 4 for examples of Delzant polytopes.

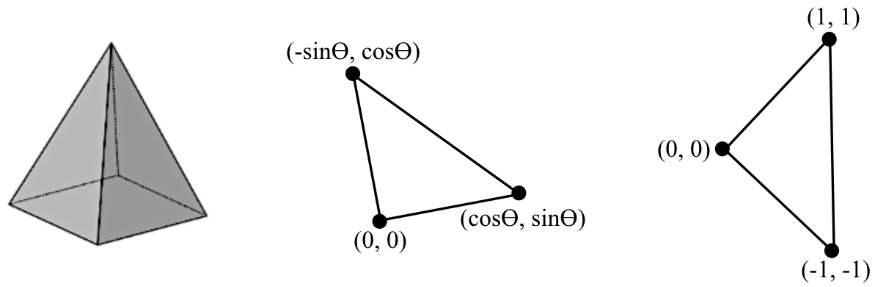

Figure 3: A square based pyramid, an isosceles right triangle slightly rotated so that the legs (if extended) never hit integer lattice points, and another isosceles right triangle: counter examples for simplicity, rationality, and smoothness, respectively.

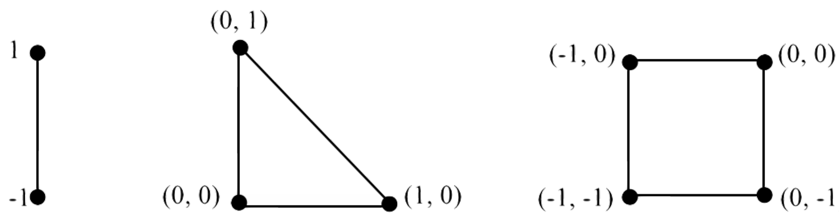

Figure 4: Delzant polytopes in one and two dimensions.

Theorem 2 (Delzant's Theorem)**.** There is a bijective correspondence between symplectic toric manifolds (up to torus-equivariant symplectomorphisms) and Delzant polytopes (up to lattice preserving mappings). This bijection is given by the moment map. Stated symbolically:

$$\{\mathbb{T}^n \rightharpoonup (M^{2n}, \omega)\}/\sim_a \overset{\mu}{\longleftrightarrow} \{\Delta\}/\sim_b$$

where $\sim_a$ is equivalence under $\mathbb{T}^n$ equivariant symplectomorphisms, and $\sim_b$ is equivalence under $SL(n, \mathbb{Z})$ mappings.

For a deeper look at theorem 2, see [6] or [2].

Remark 3. To motivate or justify the integer lattice's role in definition 7 of a Delzant polytope, note that it is related to the periodicity of the torus action on M.

Remark 4. Delzant's classification theorem allows us to study the combinatorics of Delzant polytopes in lieu of studying the properties of symplectic toric manifolds directly. Therein lies the power of theorem 2.

A Appendix

A.1 Riesz Representation Theorem

The following theorem justifies the portion of the definition for a moment map that gives the function in terms of the pairing: $\mu^X(p) = \langle\mu(p), X)\rangle$.

Theorem 3 (Riesz Representation Theorem)**.** Let H be a Hilbert space and H^* its dual space (the space of continuous linear (which is not redundant if H is not finite dimensional) functionals $H \to F$, with our field being $\mathbb{R}$ or $\mathbb{C}$). If $x \in H$, then the function φ_x defined by $\varphi_x(y) = \langle y, x\rangle$ (for all $y \in H$ and $\langle\cdot,\cdot\rangle$ the inner product of H) is an element of H^*. Moreover, every element of H^* can be written uniquely in this way.

See [4] for a proof of theorem 3.

References

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