LAB OF GEOMETRY AT MICHIGAN

PROJECT DESCRIPTIONS

Contents

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Legendrian knots associated to certain plane curves.

Faculty advisors: Caleb Ashley and Pat Boland

Grad student mentor: Alex Kapiamba

Project description. Knots arise in many mathematical areas, including geometry, topology, and number theory. Given a closed plane curve (equipped with a unit speed parametrization) we consider the curves image in the unit tangent bundle of the plane. When the curve has no self tangencies and cusps, this image is a knot. One goal of this project is to devise a conjecture for the types of knots that arise from a natural family of plane curves. To achieve this goal, we will focus much energy on ways to visualize the knots (using both paper and string models and computer graphics) and means to interpret and classify the visualizations.

Prerequisites.

Desired background:

- understanding of parametrized curves and tangent vectors (on the level of Math 116)
- understanding of matrices associated to rotations of the plane and space (on the level of Math 217)
- understanding of the geometry of complex numbers (on the level of Math 216 and 217)
- some experience with Maple, Matlab, or Mathematica

Sophisticated Background Reading: Topological Invariants of Plane Curves and Caustics. V.I. Arnold (AMS University Lecture Series, Volume 5)

The motion of point vortices on surfaces.

Faculty advisors: Dan Cianci and Joern Zimmerling

Grad student mentor: Yuxin Wang

Project description. Colloquially, we refer to a vortex as anything that causes nearby particles to follow a roughly circular trajectory. For instance, water flowing down a drain or hurricanes in the atmosphere are examples of these types of vortices. One way to try to capture this notion mathematically would be to look at the curl of a 3-dimensional vector field. Recall that the curl of a vector field measures the tendency of nearby particles traveling along the vector field to rotate. So if we think of the vector field as giving the velocity of a particle in some fluid at a point in space, then the curl measures how vortex-like the fluid is at a particular point. Indeed, in fluid mechanics the curl of a vector field is called the vorticity.

In this project we will study an idealized fluid in two dimensions where the vorticity is concentrated at certain discrete points called point vortices. These point vortices behave like particles and will follow paths determined from the configuration of other nearby point vortices (similar to how planets follow orbits depending on the configuration of nearby planets). They provide an interesting toy model for understanding fluids. The goal of this project is to computationally investigate the dynamics of these point vortices in relatively complicated geometries. For instance, what happens to their motion when the point vortices are obstructed by impenetrable islands or when their motion is constrained to curved surfaces (e.g. spheres and tori) instead of the plane? Initially, we will try to understand the equations that govern the point vortex dynamics in these geometries. Then we will focus on how to (efficiently) compute the trajectories (i.e. numerically solve the equations that govern the dynamics).

Prerequisites. Linear algebra and ODEs (Math 217 and Math 216/316 or equivalent), and coding skills (EECS 280 or knowledge in a high level programming language). A course in numerical methods (e.g. Math 371 or 471) is preferred, but not necessary.

Entropy degeneration of ideal projective pants.

Faculty advisors: Harry Bray and Giuseppe Martone

Grad student mentor: Mitul Islam

Project description. If you put two hexagons face to face in 3-space, and then only glue every other edge of the opposing hexagons together by a translation, the topological object you get is a pair of pants. These pants are the fundamental building blocks of hyperbolic surfaces (like a surface of genus 2). We can study the geometries of these surfaces by studying how the pants are glued together, and the geometries of the pants. The latter amounts to understanding the geometries of the hexagons that formed the pants to begin with.

In this project we explore a moduli space of geometries called convex projective geometries on some special pants, whose cuffs have been stretched to infinity. The actual boundaries of the cuffs are diameter zero. The goal will be to explore an extension of results of Nie and Zhang for compact surfaces to this noncompact setting. The result states that a number called the entropy, which captures the topological complexity of the geometry, can vary as much as is allowed in this moduli space of convex projective geometries. Our exploration will include creating pictures of the deformations and providing numerical evidence for the conjecture.

Prerequisites. 217, 451 or equivalent required. Any of 490/590, 433, or 412, are recommended.

Hilbert polynomials and monomial ideals.

Faculty advisors: Tim Ryan

Grad student mentors: Zhan Jiang and Monica Lewis

Project description. Given an ideal I in the ring of polynomials with complex coefficients in n variables, $R = C[x_1, \ldots, x_n]$, one can define the Hilbert function by letting $h_I(d)$ be the dimension of the space of degree d polynomials in R/I. A classical theorem in algebraic geometry states that this function eventually agrees with a polynomial called $H_I(d)$. Grothendieck proved the existence of a space parametrizing all ideals with a fixed Hilbert polynomial.

The geometry of these spaces depends greatly on the monomial ideals, e.g. (xyz, x^2) not $(x^2 + y^2)$, with the given Hilbert polynomial. In this LoG(M) project, we aim to study the monomial ideals with a given Hilbert polynomial and fixed number of variables. In particular, we will count these by hand in some examples and develop a program to compute them in every case. With this program, we will work towards developing a formula that outputs the number of monomial ideals given the number of variables and the Hilbert polynomial.

Going beyond this, it would be interesting to compute the dimension of the spaces of polynomials which limit to each monomial ideal under the action of C^* which sends $(x_1, \ldots, x_n)$ to $(ax_1, \ldots, a^n x_n)$ as these correspond to interesting geometric loci in the Hilbert scheme.

Prerequisites. Math 217, Math 412 are required. Math 465 is recommended/helpful but not required.