LOG(M) EXTREMAL VARIETIES
PROJECT DESCRIPTION

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One major element of the historical study of (algebraic) surfaces has been the study of the lines that lie on each surface. Over fields of characteristic 0, the situation is relatively well understood. Over fields of positive characteristic, there have been recent developments showing that the situation is very different. In this project, we will explore the geometry of certain surfaces with extremely large numbers of lines in positive characteristic.

To be more explicit, we will start with the case where our field has characteristic 0. Let $f \in \mathbb{R}[x, y, z]$ be a polynomial of degree d in three[1] variables. The points where f vanishes determines a surface S in $\mathbb{R}^3$, i.e.

$$S = \{\vec{x} \in \mathbb{R}^3 : f(\vec{x}) = 0\}.$$

One interesting feature of surfaces is the number of lines they contain. If $d > 3$, then for a general f, the surface S contains no lines. When S is smooth (and defined over a field of characteristic 0), the number of lines on S is bounded above by $11d^2 - 28d + 12$.

If we instead let our field be a field of positive characteristic p, (for example, think about $\mathbb{Z}/p\mathbb{Z}$), then there are surfaces that are known to wildly violate this bound. In particular, a recently studied class of surfaces, called *extremal surfaces*, are known to have $(d^2 - 3d + 3)d^2$ many lines. Extremal surfaces are very approachable as they can be defined in terms of linear algebra. These surfaces and their configurations of lines have deep connections to arithmetic geometry, combinatorics, and coding theory.

In this project, we will study the configurations of lines on extremal surfaces. More specifically, students will aim to classify the sizes of (maximal) sets of skew lines, i.e. the sizes of sets of lines where all of the lines are disjoint from one other. The team will use the algebra and geometry of the surfaces but may also use statistical techniques such as Markov Chain Monte Carlo methods to approach this problem. Time permitting, this project has several further directions. In parallel to this, we will building a code base for the current and future study of these surfaces.

Prerequisites: Math 412 or Math 493 or equivalent

[1] In reality, we will be working with four variables, because we will be working on something called *projective space*, and we sometimes prefer to think about about *algebraically closed* fields like $\mathbb{C}$ instead of $\mathbb{R}$.