Department of Mathematics
Fall 2015 Graduate Course Descriptions

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501 AIM Student Seminar Alben F 12:00-1:00pm & 3:00-4:00pm

Prerequisites: *You must be a graduate student in the AIM program to register for this course.*

Math 501 is a required course for all students enrolled in the Applied and Interdisciplinary Mathematics (AIM) MS and PhD graduate programs. In the Winter term, all first-year AIM students from both programs must sign up for this course. Due to the highly specialized content of the course, enrollment is available only for students in an AIM degree program.

The purpose of Math 501 is to address specific issues related to the process of studying applied mathematics in the AIM program and becoming an active member of the research community. The weekly meetings of the class will be divided among three types of sessions:

1. "Focus on. . . " presentations. These are presentations on various topics, some of immediate practical significance for students and others of a further-reaching nature. These discussions will include aspects of scholarly writing, research, and career development.

2. AIM Faculty Portraits. These are short presentations by faculty members in the Mathematics Department and other partner disciplines who are potential advisors or committee members for AIM students. The AIM faculty portraits provide a direct channel for students to discover what research is being done in various areas by current faculty, and to see what kind of preparation is required for participating in such research.

3. AIM Research Seminar Warm-up talks. One of the course requirements for Math 501 is weekly attendance of the AIM Research Seminar that takes place from 3-4 PM each Friday. The warm-up talks are presentations during the regular course meeting time by particularly dynamic speakers slated to speak in the AIM Research Seminar later the same day as a way to provide background material with the goal of making the AIM Research Seminar lecture more valuable for students.

Weekly attendance both of the course meeting and also of the AIM Research Seminar is required for Math 501. If you are registering for Math 501 you must be available both during the regular class time of 12-1pm on Fridays as well as during the AIM Research Seminar which runs 3-4pm on Fridays. If you are teaching, you should keep both of these obligations in mind when you submit your class/seminar schedule prior to obtaining a teaching assignment. Other requirements, including possible assignments related to topics discussed in the lectures, will be announced by the instructor in class.

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520 Life Contingencies 1 Young TTh 8:30-10:00am & 10:00 - 11:30am

Prerequisites: *Math 424 and 425 or permission from the instructor.*

Quantifying the financial impact of uncertain events is the central challenge of actuarial mathematics. The goal of this course is to teach the basic actuarial theory of mathematical models for financial uncertainties, mainly the time of death. The main topics are (1) developing probability distributions for the future lifetime random variable, and (2) using those distributions to price life insurance and annuities.

I require that you have taken Math 425, Introduction to Probability, or an equivalent calculus-based probability course. I also require that you have taken Math 424, Interest Theory. In fact, I assume that you did quite well in those courses, with grades of B+ or better.

The text for the course is {\it Actuarial Mathematics}, second edition, by Bowers, Gerber, Hickman, Jones, and Nesbitt, published by the Society of Actuaries in 1997.

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523 Risk Theory Marker TTh 11:30am-1:00pm & 1:00- 2:30pm

Prerequisites: *Math 425 with a grade of C- or better*

The goals of this course are to understand parametric distributions for the purpose of (1) modeling frequency, severity, and aggregate insurance losses, (2) analyzing the effects of insurance coverage modifications, and (3) simulating losses from those parametric distributions.

The required text for the course is Loss Models: From Data to Decisions by Stuart A. Klugman, Harry H. Panjer, and Gordon E. Willmot, fourth edition, published by John Wiley in 2012.

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525 Probability Theory Barvinok TTh 8:30 - 10:00am & TTh 10:00-11:30am

Cohen MWF 8:00 – 9:00am

This is a fairly rigorous introduction to probability theory with some emphasis given to both theory and applications, although a knowledge of measure theory is not assumed. Topics covered are: probability spaces, conditional probability, discrete and continuous random variables, generating functions, characteristic functions, random walks, Poisson process, branching processes, limit theorems.

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526 Stochastic Processes | Nadtochiy | TTh 10:00-11:30am

Prerequisites: *Math 525 or basic probability theory including: probability measures, random variables, expectations, cumulative distribution and probability density functions, conditional probabilities and independence, law of large numbers. Good understanding of advanced calculus covering limits, series, the notions of continuity, differentiation and integration.*
Recommended: Various types of convergence of random variables (almost sure, in probability, in distribution); limit theorems for sums of random variables (e.g. central limit theorem); interchanging the limit and integration/expectation (monotone and dominated convergence theorems, Fubini's theorem); linear algebra, including matrix calculus, eigenvalues and eigenfunctions.

The material is divided between discrete and continuous time processes. In both, a general theory is developed and detailed study is made of some special classes of processes and their applications. Some specific topics include: Markov chains (Markov property, recurrence and transience, stationarity, ergodicity, exit probabilities and expected exit times); exponential distribution and Poisson processes (memoryless property, thinning and superposition, compound Poisson processes); Markov processes in continuous time (generators and Kolmogorov equations, embedded Markov chains, stationary distributions and limit theorems, exit probabilities and expected exit times, Markov queues); martingales (conditional expectations, gambling (trading) with martingales, optional sampling, applications to the computation of exit probabilities and expected exit times, martingale convergence); Brownian motion (Gaussian distributions and processes, equivalent definitions of Brownian motion, invariance principle and Monte Carlo, scaling and time inversion, properties of paths, Markov property and reflection principle, applications to pricing, hedging and risk management, Brownian martingales). Significant applications will be an important feature of the course.

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550 Intro to Adaptive Dynamics | Doering | MW 8:30-10:00am

Prerequisites: *Math 215, Math 216, Math 217, and Math 425 (or equivalents)*

This course is an introduction to applications and integration of dynamical systems and game theory to model ecological and evolutionary processes. Topics include Lotka-Volterra systems, non-cooperative games, replicator dynamics and genetic mechanisms of selection and mutation, and other adaptive systems. Complex Systems 510 is the same course.

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555 Intro to Complex Variables Wu TTh 1:00-2:30pm

Prerequisites: *Courses in elementary real analysis (e.g. Math 451) and multivariable calculus (eg., Math 215 or Math 255) are essential background.*

Course Description: This course is an introduction to the analysis of complex valued functions of a complex variable with substantial attention to applications in science and engineering. Concepts, calculations, and the ability to apply principles to problems are emphasized alongside rigorous proofs of the basic results in the subject.

Topics covered include the Cauchy-Riemann equations, Taylor series, Laurent expansions, Cauchy integral formula, residues, the argument principle, harmonic functions, maximum modulus theorem, conformal mappings and applications including evaluation of improper real integrals and fluid mechanics.

Text: R. A. Silverman, Complex Analysis with Applications, Dover Publications, 1984. ISBN 0-486-64762-5

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556 Applied Functional Analysis Gilbert TTh 10:00 - 11:30am

Prerequisites: *Math 217, 419, or 420; Math 451 and Math 555.*

This is an introduction to methods of applied analysis with emphasis on Fourier analysis and partial differential equations. Students are expected to master both the proofs and applications of major results. The prerequisites include linear algebra, advanced calculus, and complex variables.
Content:
Topics in functional analysis that are used in the analysis of ordinary and partial differential equations. Metric and normed linear spaces, Banach spaces and the contraction mapping theorem, Hilbert spaces and spectral theory of compact operators, distributions and Fourier transforms, Sobolev spaces and applications to elliptic PDEs.
Alternatives:
Math 454 (Bound Val. Probs. for Part. Diff. Eq.) is an undergraduate course on the same topics.
Subsequent Courses:
Math 557 (Methods of Applied Math II), Math 558 (Ordinary Differential Equations), Math 656 (Partial Differential Equations), and Math 658 (Ordinary Differential Equations).

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558 Advanced Ordinary Differential Equations and Dynamical Systems — Simon — TTh 2:30-4:00pm

Prerequisites: *Basic Linear Algebra, Ordinary Differential Equations (Math 216), Mul- tivariable Calculus (215). Some exposure to more advanced mathematics e.g. Advanced Calculus (Math 450/451) or Advanced Mathematical Methods (Math 454).*

Differential equations model systems throughout science and engineering and display rich dynamical behavior. This course emphasizes the qualitative and geometric ideas which characterize the post-Poincaré era. It surveys a broad range of topics with emphasis on techniques, and results that are useful in applications. It is intended for students in mathematics, engineering, and the natural, biological, and social sciences. It is a core course for the Applied and Interdisciplinary Mathematics graduate program.

Outline. Chapters 1-10 + Ch. 15 of Hirsh-Smale-Devaney.
Plus online materials prepared to complement the text.
There are more complements than we will treat.

- Phase line. Dynamics in dimension 1 and 1.5. Bifurcations. Poincaré map.
- Existence, uniqueness, perturbation theory.
- Theory of constant coefficient linear systems. Spectral theorems.
- Linearization at equilibria for nonlinear systems.
- Phase plane solutions of linear and nonlinear systems.
- Stable and unstable manifolds. Conjugation of sinks/sources.
- Lyapunov functions. LaSalle's invariance principal.
- Gradient flows and Hamiltonian systems.
- Periodic solutions, stability, Poincaré map, omega-limit set, Poincaré -Bendixson and Bendixson-duLac Theorems.
- Bifurcation theory of equilibria. Pitchfork and Hopf bifurcations.
- Introduction to chaotic dynamics. Definitions and first examples.

Homework. Graded assignments weekly.
Exams. Two in class exams plus final exam.
Careful proofs will be presented in class.
Homeworks and exams will concentrate on using rather than proving.
Grading. Homework 20%, Each in-class exam 20%, Final Exam 40%.

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565 Combinatorics and Graph Theory Mustata MWF 11:00am – 12:00 PM

Prerequisites: *The students will need to be familiar with proofs, as well as with some basic notions of algebra, as covered for example in Math 412 or Math 451..*

The goal of the course is to provide an introduction to some basic notions, techniques, and results in combinatorics. The first part of the course will be devoted to graphs. We will discuss general facts about graphs, trees, colorings of graphs, extremal graphs, and tournaments. In the second part of the course we will cover lattices and Mobius inversion, projective and combinatorial geometries, designs, simplicial complexes, matroids, and hyperplane arrangements.

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571 Numerical Linear Algebra Viswanath MWF 11:00am-12:00pm

This class is a graduate level introduction to numerical linear algebra. Numerical solution of linear systems, finding eigenvalues and singular values, and solving linear least squares problems are the main topics. We will discuss condition numbers, numerical stability, QR factorization, SVD, the QR algorithm as well as iterative methods (GMRES, Arnoldi, Conjugate Gradients, Lanczos).

The homework assignments will use Python or MATLAB, with the choice left to the student, and provide a means to gain proficiency in using one of those languages. Applications to topics such as the KKT conditions and convergence of the perceptron, which is a basic machine learning algorithm, will be included.

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573 Financial Mathematics I Keller TTh 1:00-2:30pm

TBD

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591 General and Differential Topology Koch MWF10:00 - 11:00am

Prerequisites: *Math 451, Math 451, and Math 590*

This is one of the basic courses for students beginning the PhD program in mathematics. The approach is rigorous and emphasizes abstract concepts and proofs. The first 2-3 weeks of the course will be devoted to general topology, and the remainder of the course will be devoted to differential topology. Note: this course has been recently restructured and is more advanced than it was in previous years.

Topics may include: Product and quotient topology, CW-complexes, group actions, topological groups, topological manifolds, smooth manifolds, manifolds with boundary, smooth maps, partitions of unity, tangent vectors and differentials, the tangent bundle, submersions, immersions and embeddings, smooth submanifolds, Sard's Theorem, the Whitney embedding theorem, transversality, Lie groups, vector fields, Lie brackets, Lie algebra, multilinear algebra, vector bundles, differential forms, exterior derivatives, orientation, De Rham cohomology groups, homotopy invariance, degree theory.

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593 Algebra I Snowden MWF 2:00 - 3:00pm

Prerequisites: *Linear algebra and some exposure to abstract algebra at the undergraduate level. You should be comfortable with the idea of composition laws on a set (as appear in the definitions of groups and rings). Ideally, you would also be familiar with the definitions of group and rings though I will not assume you know much about them.*

This is a first graduate-level algebra course. The main topics are rings, modules, tensor products, and multi-linear algebra. We will also touch on some basic aspects of commutative algebra, homological algebra, and category theory.

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596 Analysis I (Complex) Barrett MWF 9:00-10:00 am

This course covers the Complex Analysis portion of the syllabus for the Qualifying Review Exam in Analysis for the Mathematics Doctoral Program.

Topics to be covered include:

*Complex elementary functions, conformal mapping, the Riemann sphere, linear fractional transformations, rational functions;

*Complex derivatives, Cauchy-Riemann equations;

*Contour integration, Cauchy's theorem, Cauchy-Green formula, Cauchy's integral formula and consequences, power series expansion and consequences;

*Harmonic functions, maximum principle, Dirichlet's problem;

*Isolated singularities, residues, application to computation of definite integrals, meromorphic functions, argument principle, Rouche's theorem;

*Equicontinuity, Montel's theorem, Schwartz's lemma, Riemann mapping theorem.

Homework will be assigned weekly, and midterm and final exams will be given.

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602 Real Analysis II: Functional Analysis — Bieri — MW 8:30-10:00am

Prerequisites: *Math 590 and Math 597*

Introduction to functional analysis; metric spaces, completion, Banach spaces, Hilbert spaces, L^p spaces; linear functionals, dual spaces, Riesz representation theorems; principle of uniform boundedness, closed graph theorem, Hahn-Banach theorem, Baire category theorem, applications to classical analysis.

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612 Lie Algebras — Derksen — MWF 1:00-2:00pm

This course is an introduction to the theory of Lie algebras and their representations. Lie algebras appear naturally in the study of algebraic/Lie groups. They are fundemental in Algebra and in Geometry. The study of finite dimensional Lie algebras leads to beautiful combinatorial structures such as root systems and Weyl/Coxeter groups. Topics that will be discussed include: structure theorems for Lie algebras, classifications of root systems, semi-simple Lie algebras and reflection groups, universal enveloping algebras and the Poincar´e-Birkhoff-Witt Theorem, heighest weight modules, character formulas and tensor product formulas for representations.

Text: James E. Humphreys, Introduction to Lie Algebras and Representation Theory, Graduate Texts in Mathematics, Springer. 1

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614 Commutative Algebra — Hochster — MWF 2:00-3:00pm

Prerequisites: *Math 593-594 or a strong undergraduate or graduate couse in algebra, and some basic general topology. Students with a question about prerequisites should discuss with the instructor.*

This course is an introduction to commutative algebra with emphasis on the theory of commutative Noetherian rings. One theme in the course will be to explain why every commutative ring is a geometric object. Specific topics covered will include localization, uses of the prime spectrum, integral extensions, structure of finitely generated algebras over a field (Noether normalization), Hilbert's Nullstellensatz, an introduction to affine algebraic geometry, primary decomposition, discrete valuation rings, Dedekind domains, Artin rings, dimension theory, completion, and Hilbert functions. Some category theory will be discussed. This material is particularly useful to students with interests in commutative or non-commutative algebra, algebraic geometry, several complex variables, algebraic groups or Lie theory, number theory, or algebraic combinatorics. Lecture notes will be provided.

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623 Computational Finance Bayraktar TuTh 10:00-11:30am

Prerequisites: *Differential equations (e.g. Math 316); probability theory (e.g. Math 525/526, Stat 515); numerical analysis (Math 471 or Math 472); mathematical finance (Math 423 and Math 542/IOE 552, Math 506 or permission from instructor); programming (e.g. C, Matlab, Mathematica, Java).*

This is a continuation of Math 472. This course starts with the introduction to numerical methods for solving differential equations of evolution, including the Partial Differential Equations (PDEs) of parabolic type. Convergence and stability of explicit and implicit numerical schemes is analyzed. Examples include the generalized Black- Scholes PDE for pricing European, American and Asian options. Another part of the course is concerned with the Monte Carlo methods. This includes the pseudo random number generators (with applications to option pricing) and numerical methods for solving stochastic differential equations (with applications to Stochastic Volatility models). Finally, the students are introduced to the idea of calibration, which allows one to determine the unknown model parameters from observed quantities (typically, prices of financial products). The calibration is first formulated as a general inverse problem, then, the solution methods are presented in several specific settings. The theory is accompanied by applications of proposed numerical methods in particular models of Stochastic Volatility and Interest Rate models. This includes an in-depth study of numerical methods for pricing, hedging and calibration in the Hull-White and Black-Derman-Toy models. A part of the coursework requires programming in a high-level language.

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625 Probability with Martingales Baik MW 11:30am-1:00pm

This is a rigorous introductory course on probability using measure theory. We will cover measure theory, law of large numbers, central limit theorem and random walks. If time allows, martingales and Brownian motions will also be discussed. Prior familiarity with probability at undergraduate level is assumed.

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631 Algebraic Geometry I Ho MWF 11:00am-12:00pm

Prerequisites: *Math 594 or permission of instructor. Graduate Standing.*

Theory of algebraic varieties: affine and projective varieties, dimension of varieties and subvarieties, singular points, divisors, differentials, intersections. Schemes, cohomology, curves and surfaces, varieties over the complex numbers.

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636 Topics in Differential Geometry: Dynamics and Geometry Spatzier MWF 12:00-1:00pm

Prerequisites: *Mostly standard material from alpha coureses. We will review other items as needed.*

GEOMETRY, GROUPS AND DYNAMICS

Amazingly, these three rather different subjects have a lot of interactions. Particularly wonderful are

Gromov's polynomial growth theorem: characterizes finitely generated groups with polynomially many elements of word length n.

Quasi-isometries of groups and spaces, mostly of negative or nonpositive curvature (CAT(0) or Gromov hyperbolic): quasi-isometries generalize Lipschitz maps. Amazingly many groups and spaces are completely determined by their quasi-isometry type

Random walks on groups: characterizations of different types of groups and spaces in terms of random walks or Brownian motion

Spectral gaps are related to properties of random walks. There are relations to expanders, i.e. highly connected graphs with relatively few edges, and the Kazhdan property.

Spectral gaps are also closely related to discrete subgroups of Lie groups, especially the so-called thin groups and Apollonian packings and geometrically finite subgroups of hyperbolic space.

Rigidity properties of groups and spaces: Most well-known are Mostow and Margulis rigidity theorems and generalizations to non positively curved spaces. The length of closed geodesics often determines the space (and group).

These are some highlights, close to the boundary of current research and/or fundamental to the area. I am planning to discuss some of them depending on class interest.

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656 Introduction to Partial Differential Equations Smoller MW 10:00-11:30am

Prerequisites: *Math 451, 452, Math 556 or Math 597.*

This is an introductory course. Topics will be: Characteristics and Initial-Value Problems, One-Dimensional Wave Equation, Uniqueness and Energy Integrals, Holmgren's Uniqueness Theorem, An Initial-Value Problem for a Hyperbolic Equation, Distribution Theory and Fundamental Solutions, Second-Order Linear Elliptic Equations, Second Order Parabolic Equations, Introduction to Shock Waves, Reaction-Diffusion Equations.

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665 Combinatorial Theory: Symmetric Functions — Fomin — TTh 11:30am-1:00pm

This is an introduction to the foundations of the classical theory of symmetric functions from a combinatorial perspective. Core topics include Young tableaux, Schur functions, and related combinatorial algorithms and enumeration problems. The course will conclude by a survey of applications of symmetric functions to various areas of mathematics such as linear algebra, representation theory, and enumerative geometry.

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671 Topics in Numerical Methods: Fast Algorithms — Veerapaneni — TTh 1:00-2:30pm

Prerequisites: *Numerical analysis, basic theory of ordinary and partial differential equations.*

This course will cover selected topics in fast algorithms research. Emphasis will be given on techniques that can be used for discretization and computational solution of partial differential equations. Topics will include:

- Complexity Analysis
  Review of linear & nonlinear solvers
  Krylov subspace methods

- Cartesian grid methods
  Multigrid and multiresolution methods

- Fourier spectral methods
  Nonuniform FFTs
  Butterfly Algorithms
  Ewald summation

- Numerical Methods & fast algorithms for integral equations.
  Fast Gauss Transform
  Adaptive algorithms
  Tree codes
  Fast multipole methods

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675 Analytic Theory of Numbers — Lagarias — MWF 2:00-3:00pm

Prerequisites: *It would be useful to know number theory equivalent to Math 575, one complex variable equivalent to Math 596, ability to write proofs.*

This is a first course in analytic number theory. The methods of analytic number theory are useful and applicable in many areas of mathematics. It will consider multiplicative number theory and properties of primes as a focus. It will emphasize methods. It will start with real analysis methods following Tenenbaum, arithmetic functions, prime number theory, hyperbola method, Euler-Maclaurin summation. For complex analysis methods and Dirichlet series it will follow Davenport and Montgomery. It will include basic theory of Riemann zeta function and Dirichlet L-functions to primes in arithmetic progressions. If time permits the course may include other topics, Selberg sieve, large sieve and a topic in probabilistic number theory. Grading will be based on frequent homework.

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695 Algebraic Topology — Kriz — MWF 10:00-11:00am

Prerequisites: *Some familiarity with fundamental group and homology, such as covered in Math 592 or a comparable course, is helpful.*

This course is a continuation of the first year algebraic topology course 592, but is self-contained and can be used as a first algebraic topology course by those students who have seen the basics of the fundamental group and homology. We begin with a review of homology and cohomology with coefficients, focusing on further related topics such as products, homological algebra, Tor and Ext, group (co)homology and duality theory. We then explore how these concepts lead to derived categories of spaces and modules, basic homotopy theory and also stable homotopy theory. There is no textbook but "A Concise Course of Algebraic Topology" by J.P.May (now on the author's web page) is a helpful reference. There will be course notes online. There are no exams, but approximately 4-6 HW problems are assigned and graded weekly.

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697 Teichmuller Theory and Its Generalizations — Canary — TTh 10:00am-11:30am

If S is a closed orientable surface of genus at least 2, the Teichmuller
T(S) of S is the space of isometry classes of marked hyperbolic surfaces homeomorphic to S. Equivalently, we may think of T(S) as the space of conformal classes of Riemann surfaces which are homeomorphic to S. The study of Teichmuller space arises naturally in many fields. For example, one may view Teichmuller space as the ``universal cover'' of Moduli space, since T(S) is homeomorphic to an open ball and the mapping class group acts properly discontinuously on T(S) so that we may identify its quotient with Moduli space.

We will assume only material from the alpha courses, so we will begin with a basic introduction to hyperbolic geometry. We will produce parameterizations of Teichmuller space from both geometric and complex analytic viewpoints. Moreover, we will give a topological viewpoint on the Deligne-Mumford compactification of Moduli space.

In the second portion of the course, we will focus on more advanced topics to be chosen in consultation with the students. Possible topics include:

1) The mapping class group of a surface
2) Quasifuchsian hyperbolic 3-manifolds
3) Convex projective structures on surfaces
4) The Hitchin component of representations of the fundamental group of S into PSL(n,R)
5) Anosov representations of hyperbolic groups into semi-simple Lie groups

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709 Modern Analysis I: Measure Concentration — Rudelson — TTh 10:00-11:30am

Prerequisites: *Math 625.*

Measure concentration is an area which strives to formalize a simple idea: a quantity depending on the influence of many independent random parameters is essentially constant. The simplest manifestation of this phenomenon is the Law of Large Numbers for which the relevant random quantity is the average. Yet, the Law of Large Numbers, being a limit law, may be hard to apply since it does not capture the behavior for a fixed number of samples. Measure concentration takes a different point of view, and tries to show that a relevant quantity is close to a constant with high probability for a large but fixed number of variables.

Statements of this type provide a powerful tool in many areas. In geometric functional analysis, one uses concentration to study the properties of random convex bodies, or properties of deterministic bodies by analyzing their random sections and projections. In combinatorics, concentration is the base of the probabilistic method allowing to prove the existence of certain subsets of a large structure by looking at its random pieces. Measure concentration is also a major tool used to justify the convergence of randomized algorithms in computer science.

In this course we will consider different methods of establishing concentration in both discrete and continuous setting along with the applications. Among possible applications we can discuss the Johnson-Lindenstrauss dimension reduction lemma, Dvoretzky's theorem on the existence of almost Euclidean subspaces of a general normed space, counting copies of a small subgraph in a large random graph etc.

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731 Topics in Algebraic Geometry: Perverse Sheaves — Bhatt — TTh 11:30am-1:00pm

Prerequisites: *Algebraic geometry at the level of Hartshorne's book, or equivalent. No familiarity with etale cohomology is necessary, though prior experience with basics of constructible sheaves will be quite useful.*

This class will discuss perverse sheaves, emphasizing their utility in understanding the topology of algebraic varieties. A sample topic that will be discussed is the theory of the Radon transform and its applications to Hodge theory.

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756 Advanced Topics in Partial Differential Equations: Integrable Systems and Riemann-Hilbert Problems — Miller — TTh 1:00-2:30pm

Prerequisites: *Math 454 or Math 556.*

A Hamiltonian system of ordinary differential equations on the phase space $\mathbb{R}^{2n}$ is said to be (Liouville-Arnol'd) \emph{integrable} if there exist n functionally independent constants of motion that are in involution with respect to the underlying Poisson bracket. In practical terms such dynamical systems can be regarded as explicitly solvable, because there exists a change of coordinates that linearizes the dynamics. This is a classical notion.

In the late 1960's it was discovered that some initial-value problems for nonlinear partial differential equations (PDE) describing wave propagation (e.g., the Korteweg-de Vries equation and the cubic nonlinear Schr\"odinger equation) can be seen as infinite-dimensional analogues of classical integrable Hamiltonian systems, and a new technique was advanced for constructing the change of coordinates that trivializes the dynamics for these problems. This technique immediately yields a huge variety of exact solutions (\emph{solitons} and their relatives) and is based on a nonlinear analogue of the Fourier transform (the latter of course trivializes the dynamics of linear constant-coefficient equations on $\mathbb{R}^n$) now called the \emph{scattering transform}. The \emph{direct scattering transform} considers the solution of the PDE at hand at a fixed time t to be a coefficient in some linear differential equation with a complex spectral parameter λ and maps it to a set of λ-dependent scattering data. The \emph{inverse scattering transform} has to reconstruct the coefficient in the scattering equation from its scattering data. The latter is an inverse problem of some interest in many other areas of applied mathematics (e.g., tomography, remote sensing) as well.

Because appropriately selected solutions of the direct-scattering equation frequently depend analytically on λ in some domains of $\mathbb{C}$, the inverse scattering transform can often be formulated as a \emph{Riemann-Hilbert problem}, in which a function analytic in various domains of the complex plane has to be reconstructed from ``jump conditions'' that relate the traces on curves that bound adjacent domains. Such a problem can in turn be recast as a system of singular integral equations with Cauchy-type kernels.

Riemann-Hilbert problems have other applications as well. They can be used to characterize polynomials orthogonal with respect to arbitrary weights, and this leads to a useful representation of various statistics of random matrices. There is also a class of integral operators that can be inverted with the help of an associated Riemann-Hilbert problem.

Finally, there is an analogue of the classical method of steepest descent for the asymptotic expansion of integrals that applies to certain Riemann-Hilbert problems. This method allows important information to be gleaned from Riemann-Hilbert problems in which a large or small parameter appears in the jump conditions. This method has been used to analyze solutions of integrable PDE with extreme precision in both the large time and semiclassical (or weakly-dispersive) limits. It has also been used to obtain large-degree asymptotics for general orthogonal polynomials and to prove universality conjectures in random matrix theory.

This course will be an introduction to these and other related topics. Complex analysis (Math 555 or Math 596) is an essential prerequisite. Students who have taken Asymptotic Analysis (Math 557) will more easily grasp and appreciate the steepest descent method for Riemann-Hilbert problems, but this is not as important of a prerequisite. While much of the course will involve solutions of nonlinear PDE, the relevant background required is more along the lines of Math 454 or Math 556 than of Math 656 or Math 657. There is no textbook, and materials will be distributed by the instructor as the course progresses. Students will be evaluated on the basis of some homework sets and possibly a term project culminating in a presentation to the class on a topic related to the course.