# Branching Diffusions Representation for Nonlinear PDEs

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# Outline

# From linear representation (ii) to nonlinear

- Consider KPP equations

$$\text{(KPP)} \quad \partial_t v + \tfrac{1}{2}\Delta v + \beta\Big(\sum_{i=1}^n p_i v^i - v\Big) = 0$$

with $p_i > 0$ and $\sum_{k=1}^n p_i = 1$

- Branching diffusions representation :

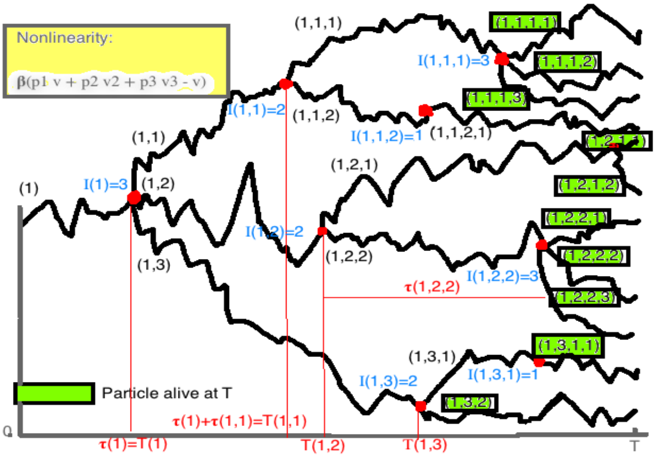
$$v(0,x) \quad = \quad \mathbb{E}\Big[\prod_{k\in\mathcal{K}_T} g(Z_T^k)\Big], \quad \text{where} \quad Z^k : \; k-\text{th particle}$$

and

$$\mathcal{K}_t \quad := \quad \big\{\text{All particles alive at time } t\big\}$$

[Skorokhod, Watanabe, McKean]

# Branching diffusion ($n = 3$)



# Outline

# Generalized KPP equation

Let $a_i(t,x)$ be bounded functions, and consider the PDE

$$\partial_t v + \mu(t,x)\cdot Dv + \frac{1}{2}\sigma^2(t,x) : D^2 v + \sum_{i=0}^{n} p_i a_i(t,x) v^i = 0$$

$$v(T,.) = g$$

Introduce the branching diffusion :

- $(\tau_k)_k$ iid $\sim \rho$, and $T_k := T \wedge (\tau_1 + \ldots \tau_k)$ : branching times
- $(I_k)_k$ iid Multinomial$(p_0, \ldots, p_n)$ : number of decendents
- Particle $k$ dies out at the branching event $T_k$, and $I_k$ **independent** particles defined by

$$dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dW_t, \quad W \text{ Brownian motion}$$

# The branching diffusion representation

Recall $\mathcal{K}_T := \{\text{particles present at } T\}$
$\overline{\mathcal{K}}_T := \cup_{t \le T} \mathcal{K}_t$ : all particles

**Theorem (Henry-Labordère, Tan & NT SPA '14)**

*Let $\rho > 0$ density on $(0, \infty)$, and $\bar{\rho}(t) := \int_t^\infty \rho(s)ds$. Then*

$$v(0,x) = \mathbb{E}_{0,x}\Big[\prod_{k \in \mathcal{K}_T} \frac{g(Z_T^k)}{\bar{\rho}(\Delta T)} \prod_{k \in \overline{\mathcal{K}}_T \setminus \mathcal{K}_T} \frac{a_{I_k}(T_k, Z_{T_k}^k)}{\rho(\Delta T_k)}\Big]$$

*Moreover, this representation extends to the path-dependent case*

- Numerical implications
- In the rest of he talk : extension to more general nonlinearities

# Sketch of proof ($\mu \equiv 0$, $\sigma \equiv 1$, so $X =$BM)

Feynman-Kac's formula $\equiv$ Duhamel's representation (with Heat kernel as fundamental solution of Heat equation)

$$v(0,x) = \mathbb{E}_{0,x}\Big[ \quad g(X_T) + \int_0^T \sum_{i=1}^n p_i a_i(t,X_t) v(t,X_t)^i \quad \Big]$$

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$$= \mathbb{E}_{0,x}\Big[\mathbb{1}_{\tau_1 > T}\frac{g(X_T)}{\bar\rho(T)} + \mathbb{1}_{\tau_1 \le T}\frac{a_{I_1}(\tau_1, X_{\tau_1})}{\rho(\tau_1)} v(\tau_1, X_{\tau_1})^{I_1}\Big] =: \mathbb{E}_{0,x}[\phi(\tau_1, X_{\tau_1})]$$

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Continue with the nonlinear term to get rid of the regression :

$$v(\tau_1, X_{\tau_1})^{I_1} = \mathbb{E}_{\tau_1, X_{\tau_1}}\big[\phi(\tau_2, X_{\tau_2}, I_2)\big]^{I_1}$$

$$= \underbrace{\mathbb{E}_{\tau_1, X_{\tau_1}}\big[\phi(\tau_2, X_{\tau_2}, I_2)\big] \times \ldots \times \mathbb{E}_{\tau_1, X_{\tau_1}}\big[\phi(\tau_2, X_{\tau_2}, I_2)\big]}_{I_1 \text{ times}}$$

$$= \mathbb{E}_{\tau_1, X_{\tau_1}}\big[\phi(\tau_{1,1}, Z^{1,1}_{\tau_{1,1}}, I_{1,1}) \times \ldots \times \phi(\tau_{1,I}, Z_{\tau_{1,I}}, I_{1,I})\big]$$

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# A class of semilinear PDEs with polynomial nonlinearity

Consider the PDE (unit diffusion for simplicity)

$$\partial_t u + \frac{1}{2}\Delta u + f(t,x,u,Du) = 0, \quad u_T = g$$

with nonlinearity (e.g. Burger's equation in 1d $uDu$)

$$f(t,x,y,z) \;=\; \sum_{(\ell_i)_{0\le i\le n}\in L} p_\ell c_\ell(t,x) y^{\ell_0} \prod_{i=1}^{n} \big(b_i(t,x)\cdot z\big)^{\ell_i}$$

- $L$ finite subset of $\mathbb{N}^{n+1}$
- $p_\ell > 0$ with $\sum_{\ell\in L} p_\ell = 1$
- $b_i(t,x)$ bounded functions

# Illustration in the context of the Burger's equation

By the Feynman-Kac formula

$$v(0,x) = \mathbb{E}_{0,x}\Big[\frac{\bar\rho(T)}{\bar\rho(T)}g(X_T) + \int_0^T (vDv)(t,X_t)\frac{\rho(t)}{\rho(t)}dt\Big]$$

$$= \mathbb{E}_{0,x}\Big[\mathbb{1}_{\tau_1>T}\frac{g(X_T)}{\bar\rho(T)} + \mathbb{1}_{\tau_1\le T}\frac{(vDv)(\tau_1,X_{\tau_1})}{\rho(\tau_1)}\Big] =: \mathbb{E}_{0,x}\big[\phi(\tau_1,X_{\tau_1})\big]$$

Continue with the nonlinear term to get rid of the regression :

$$\begin{aligned}(vDv)(\tau_1,X_{\tau_1}) &= \mathbb{E}_{\tau_1,X_{\tau_1}}\big[\phi(\tau_2,X_{\tau_2})\big]\partial_x\mathbb{E}_{\tau_1,X_{\tau_1}}\big[\phi(\tau_2,X_{\tau_2})\big]\\ &= \mathbb{E}_{\tau_1,X_{\tau_1}}\big[\phi(\tau_2,X_{\tau_2})\big]\mathbb{E}_{\tau_1,X_{\tau_1}}\Big[\frac{W_{\tau_2}-W_{\tau_1}}{\tau_2-\tau_1}\phi(\tau_2,X_{\tau_2})\Big]\\ &= \mathbb{E}_{\tau_1,X_{\tau_1}}\Big[\phi(\tau_{1,1},Z^{1,1}_{\tau_{1,1}})\frac{W_{\tau_2}-W_{\tau_1}}{\tau_2-\tau_1}\phi(\tau_{1,2},Z_{\tau_{1,2}})\Big]\end{aligned}$$

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\end{aligned}$$

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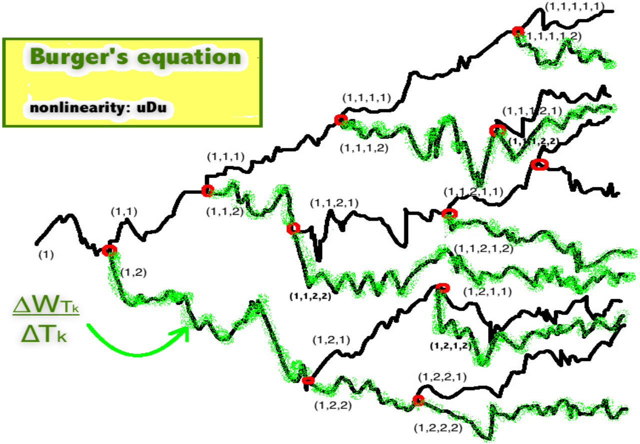
# Branching diffusion for the Burger equation

**Example :** Burgers equation $d = 1$ and $f(t, x, u, u_x) = u\, u_x$



# Likelihood ratio (Malliavin) automatic differentiation

For simplicity, $d = 1$, constant $\mu$, $\sigma = 1$ : $X_h = x + \mu h + W_h$

$$\partial_x \mathbb{E}[\phi(X_h)] \;=\; \mathbb{E}\left[\phi(X_h)\frac{W_h}{h}\right]$$

Direct integration by parts

$$\begin{aligned}
\partial_x \mathbb{E}\big[\phi(X_h)\big] = \mathbb{E}\big[\phi_x(X_h)\big] &= \int \phi_x(x+y)\frac{e^{\frac{-1}{2h}(y-\mu h)^2}}{\sqrt{2\pi h}}dy \\
&= \int \phi(x+y)\frac{(y-\mu h)}{h}\frac{e^{\frac{-1}{2h}(y-\mu h)^2}}{\sqrt{2\pi h}}dy \\
&= \mathbb{E}\left[\phi(X_h)\frac{W_h}{h}\right]
\end{aligned}$$

# Marked branching mechanism

- $(\tau_k)_k$ iid arrival times, $T_k := \tau_k \wedge T$
- $(I_k)_k$ iid Multinomial$\big(\ell, p_\ell, \ell \in L\big)$
- If $T_1 < T$ : particle dies out at $T_1$
  - $I_1 = \ell = (\ell_0, \ldots, \ell_n)$ with probability $p_\ell$
  - birth of new particles :

$$\ell_i \text{ particles of type } i, \; i = 0, \ldots, n$$

- $\mathcal{K}_t = \{k \text{ alive at time } t\}$, $\overline{\mathcal{K}}_t := \cup_{s \le t} \mathcal{K}_s$, for all $k \in \overline{\mathcal{K}}_T$ :
  - $D(k)$ its type
  - $k-$ its parent particle

  $\Longrightarrow$ Particle $k$ lives between $T_{k-}$ and $T_k$

# Using Malliavin automatic differentiation

- Malliavin automatic differentiation :

$$\mathcal{W}_k := \mathbb{1}_{\{D(k)=0\}} + \mathbb{1}_{\{D(k)\neq 0\}} b_{D(k)}(T_{k-}, X^k_{T_{k-}}) \cdot \frac{\Delta W_{T_k}}{\Delta T_k}$$

The limiting random variable is :

$$\psi := \prod_{k \in \mathcal{K}_T} \bar{\rho}(\Delta T_k)^{-1} \left[ g(X^k_T) - \mathbb{1}_{\{D_k \neq 0\}} g(X^k_{T_{k-}}) \right] \mathcal{W}_k \times \prod_{k' \in \overline{\mathcal{K}}_T \setminus \mathcal{K}_T} \left[ \rho(\Delta T_{k'}) \right]^{-1} b_{I_{k'}}(T_{k'}, X^{k'}_{T_{k'}}) \mathcal{W}_{k'}$$

# The main representation result

For independent BM $W$, $\tau \sim \rho$, and $T_1 := \tau \wedge T$, define :

$$A_p := \max_{\ell} \frac{|g|_\infty^p \vee \|W_{T_1}\|^p \|b_\ell \cdot \frac{W_{T_1}}{T_1}\|^p}{\bar{\rho}(T)^{p-1}}$$

$$B_p := \max_{\ell} \|\Delta\tau\|^{p/2} \Big\| b_\ell \cdot \frac{W_{T_1}}{T_1} \Big\|^p \Big[ |b_\ell|_\infty \sup_{t \le T} \frac{t^{-\frac{p}{2(p-1)}}}{\rho(t)} \Big]^{p-1}$$

**Theorem (Henry-Labordère, Oudjane, Tan, NT, Warin '16)**

*Assume that $g$ Lipschitz and, for some $p > 1$,*

$$\int_{A_p}^{\infty} \Big[ B_p \sum |b_\ell|_\infty |x|^{|\ell|} \Big]^{-1} dx \quad > \quad T$$

*Then $v(0,x) = \mathbb{E}_{0,x}[\psi]$, and $\psi \in \mathbb{L}^p$*

# A numerical example of dimension $d = 20$

$u(t,x) = \cos(x_1 + \cdots + x_d)\exp(\alpha(T-t))$ is solution of semilinear PDE

$$\partial_t u + \frac{1}{2}\Delta u + c\, u(b_1 \cdot Du) + b_0 \;=\; 0.$$

- Let $\alpha = 0.2$, $c = 0.15$, $b_1 = (1+\frac{1}{d}, 1+\frac{2}{d}, \cdots, 2)$.

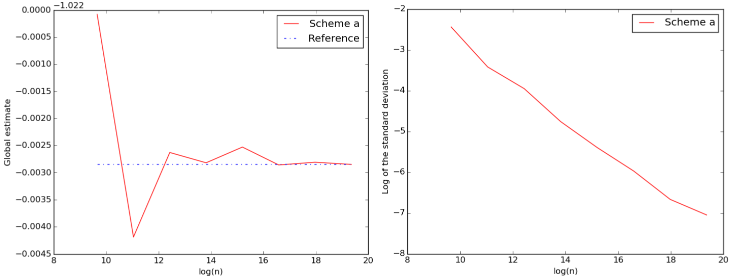



Figure – Estimation and standard deviation observed in dimension $d = 20$ depending on the log of the number of simulation used.

# Monte Carlo approximation of nonlinear PDEs

Fully nonlinear PDEs... (e.g. HJB equations)

If $T_1 < T$ : particle dies out, and is replaced with probability $p_\ell$ by

- $i_\ell$ particles of type 0
- $j_\ell$ particles of type 1 $\Longrightarrow$ first order differentiation weight
- $h_\ell$ particles of type 2 $\Longrightarrow$ second order differentiation weight

Automatic differentiation for particles $k$ of type $D(k) = 2$ :

$$\frac{\Delta W_T^2 - \Delta T}{(\Delta T)^2} \quad !!$$

# Outline

# Linear initial value problems with constant coefficients

Let $D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$, and consider the general PDE :

$$\sum_{n=1}^N a_n \partial_t^n u - \sum_{\alpha \in \mathbb{N}_M^d} b_\alpha \cdot D^\alpha u \;=\; F \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^d,$$
$$\partial_t^{n-1} u(0,.) = p_n f_n \quad \text{on } \mathbb{R}^d, \quad n = 1, \dots, N$$

where $p_n > 0$, $\sum_{n=1}^N p_n = 1$,

$f_n : \mathbb{R}^d \longrightarrow \mathbb{R}$, $n \le N$ and $F : \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow \mathbb{R}$, bounded continuous

**Examples :** Heat equation ++

- Wave equation : $\partial_t^2 u - \Delta u = F$
- Airy equation : $\partial_t u - \sum_{|\alpha|=3} b_\alpha D^\alpha u = F$... to KdV equation
- Dynamic Beam equation $\partial_t^2 u - \sum_{|\alpha|=4} b_\alpha D^\alpha u = F$
- Schroödinger equation : $i\partial_t u + \Delta u = F$... focalizing/defocalizing

# Duhamel's Representation

For all $t \geq 0$, and $x \in \mathbb{R}^d$, we have

$$u(t,x) = \sum_{n=1}^{N} \int p_n f_n(y) g_n(t, x-y) dy + \int_0^t \int F(t-s, y) g_N(t-s, x-y) dy\, ds$$

$$g_n(t,.) := \mathfrak{F}^{-1} \hat{g}_n(t,.)$$

$$\hat{g}_n(t,.) := \left\{ e^{tB(\xi)^{\mathrm{T}}} \right\}_{1,.}$$

$$b(\xi) := \textstyle\sum_{\alpha \in \mathbb{N}_M^d} i^{|\alpha|} b_\alpha \xi^\alpha,$$

$$B(\xi) := \left( \begin{array}{c|ccc} 0 & & & \\ | & & I_{N-1} & \\ 0 & & & \\ \hline b(\xi) & -a_1 & \cdots & -a_{N-1} \end{array} \right)$$

# Examples

Airy equation : $g_1(t,z) = (3t)^{\frac{-d}{3}} \mathrm{Ai}_b\left(\frac{z}{(3t)^{1/3}}\right)$ in terms of Airy function $\mathrm{Ai}_b(x) := \pi^{-d}\int_{\mathbb{R}^d_+} \cos\left(\frac{-b(\xi)}{3} + x\cdot\xi\right)d\xi$. Here $\int g_1(t,.) = \infty$ !

Wave equation : $g_1(t,z) = \partial_t g_2(t,z)$, and

$$g_2(t,z)dz = \begin{cases} \frac{1}{2r}1_{\{|z|<t\}}dz & \text{for } d=1 \\ \frac{1}{2\pi\sqrt{t^2-|z|^2}}1_{\{|z|<t\}}dz & \text{for } d=2 \\ \frac{\sigma_t(dz)}{4\pi t} & \text{for } d=3 \end{cases}$$

$\sigma_t(dz)$ : surface area on $\partial B(0,r)$

Beam equation $(u_{tt} + \partial_x^4 u = 0)$

$$g_1(r,z) = \partial_r g_2(r,z) \quad \text{and} \quad g_2(r,z) = r^{\frac{1}{2}} G\left(r^{\frac{-1}{2}} z\right),$$

where $G(x) = \frac{1}{\sqrt{2\pi}} + \frac{1}{2}\int_0^x \left[\mathrm{F}_s\left(\frac{z}{\sqrt{2\pi}}\right) - \mathrm{F}_c\left(\frac{z}{\sqrt{2\pi}}\right)\right]dz$ with the Fresnel integrals $\mathrm{F}_c(x) := \int_0^x \cos(\pi t^2/2)dt$, $\mathrm{F}_s(x) := \int_0^x \sin(\pi t^2/2)dt$

# Probabilistic representation

Assume $G_n(t) := \int \big|g_n(t,y)\big| dy < \infty$ for all $t \geq 0$, $1 \leq j \leq n$

Set $\gamma_n(t,y) := \mathrm{sg}\big\{g_n(t,y)\big\} G_n(t)$, $\quad n = 1, \ldots, N$, then :

$$u(t,x) = \mathbb{E}\Big[\mathbb{1}_{\{\tau \geq t\}} \frac{\gamma_I(t, Z^I_t)}{\bar{\rho}(t)} f_I\big(X^I_\tau\big) + \mathbb{1}_{\{\tau < t\}} \frac{\gamma_N(\tau, Z^N_\tau)}{\rho(\tau)} F\big(t-\tau, X^N_\tau\big)\Big]$$

$I, \tau, X^n_t := x - Z^n_t, 1 \leq n \leq N$ independent with

$$\mathbb{P}[I = n] = p_n, \;\; n = 1, \ldots, N$$
$$\mathbb{P}[\tau \in dt] = \rho(t) \mathbb{1}_{\{t \geq 0\}} dt, \;\; \rho > 0$$
$$\mathbb{P}\big[Z^n_t \in dz\big] = \frac{|g_n(t,z)|}{G_n(t)} dz$$

# Representation of semilinear initial value problems

## Theorem (Henry-Labordère & NT)

*Under some conditions, we have*

$$\xi_{t,x} \; := \; \prod_{k\in\mathcal{K}_t} \frac{\gamma_{I_k^t}\left(\Delta T_k^t, Z^k_{T_k^t}\right)}{\bar\rho\left(\Delta T_k^t\right)} f_{I_k^t}\left(X_t^k\right) \times \prod_{k\in\overline{\mathcal{K}}_t\setminus\mathcal{K}_t} \frac{\gamma_N\left(\Delta T_k^t, Z^k_{T_k^t}\right)}{\rho\left(\Delta T_k^t\right)} c_{J_k}\left(t - T_k^t, X^k_{T_k^t}\right) \;\in\; \mathbb{L}^1$$

*and* $u(t,x) := \mathbb{E}\left[\xi_{t,x}\right]$ *is the unique solution of the semilinear IVP*

$$\sum_{n=1}^N a_n \partial_t^n u - \sum_{\alpha\in\mathbb{N}^d_M} b_\alpha D^\alpha u = \sum_{j\geq 1} q_j\, c_j\, u^j \;\; on \; \mathbb{R}_+\times\mathbb{R}^d$$
$$\partial_t^{n-1} u(0,.) = f_n \text{ on } \mathbb{R}^d, \;\; n = 1,\ldots,N$$

Compare to Bachtin & Mueller 2010 (semilinear wave equation)

# Automatic differentiation

Define the likelihood ratio (in distribution sense)

$$w_n(t,z) := \frac{Dg_n}{g_n}(t,z), \quad (t,z) \in \mathbb{R}_+ \times \mathbb{R}^d$$

and set for all $k \in \overline{\mathcal{K}}_t$

$$\mathcal{W}_k := \mathbb{1}_{\{\theta_k^t = 0\}} + \mathbb{1}_{\{\theta_k^t \neq 0\}} \vec{b}_{\theta_k^t}\left(t - T_{k-}^t, X_{T_{k-}^t}^{k-}\right) \cdot w_{I_k^t}\left(\Delta T_k^t, Z_{T_k^t}^k\right)$$

where $\theta_k^t$ is the type of particle $k$

# More general semilinear initial value problems

## Theorem (Henry-Labordère & NT)

*Under some conditions, we have*

$$\hat{\xi}_{t,x} \;:=\; \prod_{k\in\mathcal{K}_t} \mathcal{W}_k \frac{\gamma_{I_k^t}\big(\Delta T_k^t, Z^k_{T_k^t}\big)}{\bar{\rho}\big(\Delta T_k^t\big)} \Big[f_{I_k^t}\big(X_t^k\big) - \mathbb{1}_{\{\theta_k^t\neq 0\}} f_{I_k^t}\big(X^k_{T_{k-}}\big)\Big]$$

$$\times \prod_{k\in\overline{\mathcal{K}}_t\setminus\mathcal{K}_t} \mathcal{W}_k \frac{\gamma_N\big(\Delta T_k^t, Z^k_{T_k^t}\big)}{\rho\big(\Delta T_k^t\big)} c_{J_k}\big(t - T_k^t, X^k_{T_k^t}\big)$$

*and* $u(t,x) := \mathbb{E}\big[\xi_{t,x}\big]$ *is the unique solution of the semilinear IVP*

$$\sum_{n=1}^N a_n \partial_t^n u - \sum_{\alpha\in\mathbb{N}_M^d} b_\alpha D^\alpha u + \sum_{j\geq 0} q_j\, c_j\, u^{\ell_{j,0}} \prod_{h=1}^H (b_h\cdot Du)^{\ell_{j,h}} = 0$$

$$\partial_t^{n-1} u(0,.) = p_n f_n \quad \text{on } \mathbb{R}^d, \quad n = 1,\ldots,N$$

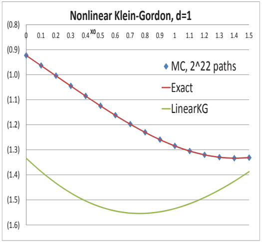
# Example : nonlinear wave equation 1

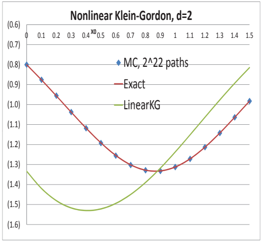
Klein-Gordon equation in $d = 1, \ldots, 3$ :

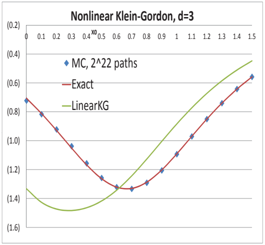
$$(\partial_{tt} - \Delta)u + u^3 + u^2 = 0, \quad u(0,x) = f_1(x), \quad \partial_t u(0,x) = f_2(x)$$

with $f_1(x) := \frac{-12}{9+2(\sum_{i=1}^d x_i)^2}$, and $f_2(x) := -\sqrt{d+1}\frac{48(\sum_{i=1}^d x_i)}{(2(\sum_{i=1}^d x_i)^2+9)^2} \Longrightarrow$

explicit solution : $u(t,x) = \frac{-12}{9+2(\sqrt{d+1}t - \sum_{i=1}^d x_i)^2}$



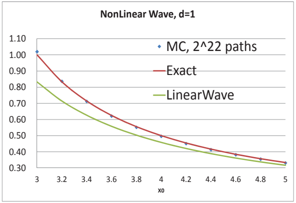
# Example : nonlinear wave equation 2

Simplified Yang-Mills equation

$$\partial_{tt}u - \Delta u + u^3 + \sum_i^d u\partial_{x_i}u = 0, \quad u(0,x) = f_1(x), \quad \partial_t u(0,x) = f_2(x)$$

with $f_1(x) = -\frac{d}{d-\sum_{i=1}^d x_i}$, $f_2(x) = \sqrt{d}\frac{d}{(d-\sum_{i=1}^d x_i)^2}$

$\Longrightarrow$ explicit solution : $u(t,x) = -\frac{d}{d+\sqrt{d}t-\sum_{i=1}^d x_i}$



# Example : nonlinear Beam equation in 1d

$$\partial_t^2 u + \partial_x^3 u + u^2 + h(t,x) = 0, \quad u(0,x) = \tanh(x), \quad \partial_t u(0,x) = \cosh(x)^{-2}$$

$\Longrightarrow$ explicit solution is $u(t,x) = \tanh(x+t)$ (for suitable choice of $h$)

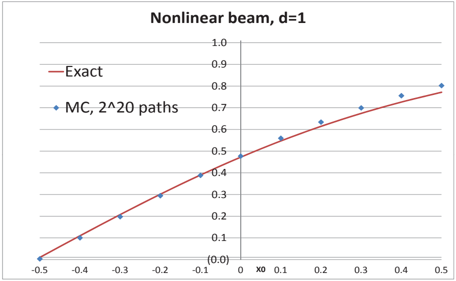



Figure – Numerical solution of nonlinear beam PDE $t = 0.5$.

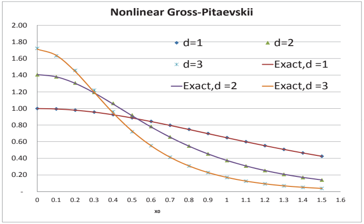
# Example : Gross-Pitaevskii PDE

For a constant $h(=-1$ in our numerics), consider

$$i\partial_t u(t,x) = -\frac{1}{2}\Delta u(t,x) + h|u(t,x)|^2 u(t,x), \quad x \in \mathbb{R}^d$$

$$u(0,x) = f_1(x) := \sqrt{d}\Big[\cosh\big(\textstyle\sum_{i=1}^d x_i\big)\Big]^{-1}$$

$$\Longrightarrow \text{Explicit solution } u(t,x) = e^{\frac{idt}{2}}\sqrt{d}\Big[\cosh\big(\textstyle\sum_{i=1}^d x_i\big)\Big]^{-1}$$



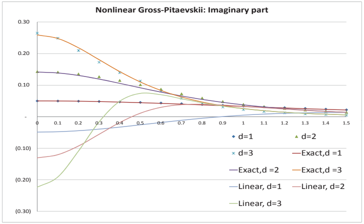




Figure – $Re(u(t,x_0))$ (left) and $Im(u(t,x_0))$ (right) for $d=1,2,3$, $t=0.1$

# Concluding remarks

• Bounded space domain (including elliptic PDEs) : serious difficulties for gradient nonlinearity, Agarwal & Claisse '17

• Variable coefficients :

- parabolic equations
- more general Cauchy problems...

• Exploiding representation for wellposed equation !

- backward iteration over "short time steps," Bouchard, Tan & Zhou '17
- Beyond power nonlinearity