Painlevé Equations: Analysis and Applications

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Outline of the talk

- Part I – Location of poles for the Hastings-McLeod solution to the Painlevé II equation

  Joint work with Min Huang and Shuai-Xia Xu

- Part II – Gap probability at the hard edge for random matrix ensembles with pole singularities in the potential

  Joint work with Dan Dai and Shuai-Xia Xu

Full list of Painlevé equations

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 6w^2 + z$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}$$

Painlevé VI . . .

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  which possess the Painlevé property
- The solutions of Painlevé equations are called the Painlevé transcendents

n-truncated solutions

- The first two Painlevé equations

$$y'' = 6y^2 + x, \qquad y'' = 2y^3 + xy + \alpha$$

  - All solutions are meromorphic
  - $x = \infty$ is the only essential singularity

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- Existence of solutions which have no lines of poles near infinity near n ($n = 1, 2, 3$) of the critical rays – n-truncated solutions

[Boutroux, 1913&1914]

n-truncated solutions

- Critical rays:

$$\Gamma_k := \left\{ x \;\middle|\; \arg x = \frac{2k\pi}{N} \right\}, \qquad k = 0, 1, \ldots, N-1,$$

where

$$N = \begin{cases} 5, & \text{for PI} \\ 6, & \text{for PII} \end{cases}$$

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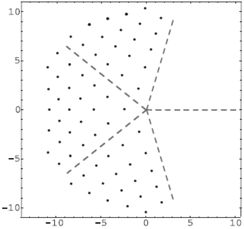
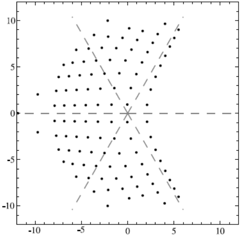
- Examples:





A conjecture of Novokshenov

Conjecture (Novokshenov, '14)

If the 2- or 3-truncated solution of Painlevé equation has no pole at infinity in a sector Ξ_k, then it has no poles in the whole sector Ξ_k, where

$$\Xi_k := \left\{ x \;\middle|\; \frac{2k\pi}{N} < \arg x < \frac{2(k+1)\pi}{N} \right\}, \qquad k = 0, 1, \ldots, N-1.$$

- Numerical confirmations:

[Fornberg-Weideman, '11&'14; Novokshenov '09]

A conjecture of Novokshenov

- For the 3-truncated solutions (tritronquées) of PI: Dubrovin's conjecture

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- The tritronquée solution describes the asymptotic behavior of solutions to the focusing NLS equation near the critical point. Poles of the tritronquées are related to the spikes of NLS solutions

  [Bertola-Tovbis, '13]

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- Dubrovin's conjecture has been completely proved recently

  [Costin-Huang-Tanveer, '14]

Hastings-McLeod solution of PII

- The Hastings-McLeod solution y_{HM} is a special solution of

$$y'' = 2y^3 + xy$$

with the asymptotics

$$y_{HM}(x) \sim \begin{cases} \mathrm{Ai}(x), & \text{as } x \to +\infty \\ \sqrt{-x/2}, & \text{as } x \to -\infty \end{cases}$$

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- The solution y_{HM} is known to be pole-free on the real axis [Hastings-McLeod, '80]

Tracy-Widom distribution in RMT

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- For the largest eigenvalue λ_{max} of an $n \times n$ GUE matrix, the random variable

$$n^{1/6}(\lambda_{max} - 2\sqrt{n})$$

converges in distribution to the well-known Tracy-Widom distribution $F_2(s)$ as $n \to \infty$

[Tracy-Widom, '94]

- Tracy-Widom distribution is universal
  - random permutation

    [Baik-Deift-Johansson, '99]
  - Asymmetric Simple Exclusion Process (ASEP) with step initial condition

    [Johansson, '00; Tracy-Widom, '09]
  - ······

Tracy-Widom distribution and y_{HM}

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Tracy-Widom distribution and y_{HM}

- There are two formulas for the Tracy-Widom distribution
  - Fredholm determinant representation:

$$F_2(s) = \det(I - A_s)$$

where A_s is the integral operator acting on $L^2(s, \infty)$ with kernel given in terms of Airy functions Ai by

$$\frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x - y} \qquad \text{Airy kernel}$$

  - Integral representation:

$$F_2(s) = \exp\left(-\int_s^\infty (x - s) y_{HM}^2(x)\, \mathrm{d}x\right)$$

The Hastings-MeLeod solution and its poles

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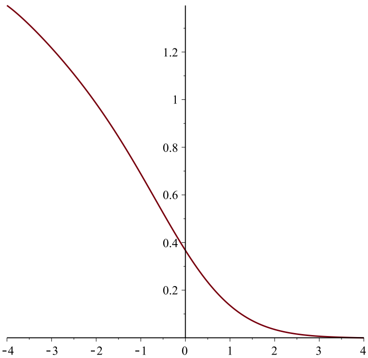

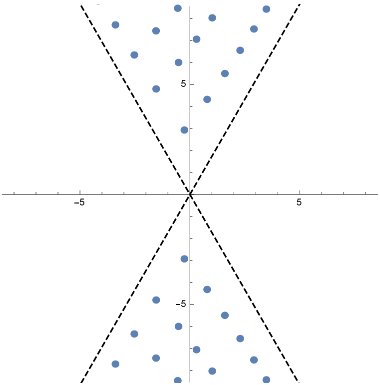

Main result

Theorem (Huang-Xu-LZ, Constr. Approx., '16)

The Hastings-McLeod solution y_{HM} of the second homogeneous Painlevé equation

$$y'' = 2y^3 + xy$$

is pole-free in the region $\arg x \in [-\frac{\pi}{3}, \frac{\pi}{3}] \cup [\frac{2\pi}{3}, \frac{4\pi}{3}]$.

Known results

- For $|x|$ large enough – Riemann-Hilbert approach

  [Its-Kapaev, '03]

- For $\arg x \in [-\frac{\pi}{3}, \frac{\pi}{3}]$ – an operator-norm estimate

  [Bertola, '12]

Strategy of proof

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- Construct an explicit quasi-solution and show the difference between real solution and the quasi-solution is small in a suitable norm
  - Difficulty: an effective quasi-solution approximation with sufficient accuracy for both small and large $|x|$

Strategy of proof

- A direct analysis based on the idea in the proof of Dubrovin's conjecture
- Construct an explicit quasi-solution and show the difference between real solution and the quasi-solution is small in a suitable norm
  - Difficulty: an effective quasi-solution approximation with sufficient accuracy for both small and large $|x|$
- Can be applied to other equations including the general PII equation with $\alpha \neq 0$

Strategy of proof

- Focus on the sector

$$\Omega := \left\{ x \in \mathbb{C} \,\middle|\, 2\pi/3 \le \arg x \le \pi \right\}$$

- Analyze y_{HM} in two regions

$$\Omega_0 := \left\{ x \in \mathbb{C} \,\middle|\, |x| \geqslant \frac{3^{4/3}}{2}, \ \ 2\pi/3 \leqslant \arg x \leqslant \pi \right\}$$

and

$$\Omega_2 := \left\{ x \in \mathbb{C} \,\middle|\, |x| \leqslant 9/4, \ \ 2\pi/3 \leqslant \arg x \leqslant \pi \right\}.$$

- Note that $\Omega \subseteq \Omega_0 \cup \Omega_2$

Properties of y_{HM}

Proposition (Its-Kapaev, '03)

Let y_{HM} be the Hastings-McLeod solution of the second Painlevé equation, then

$$y_{HM}(x) = \frac{1}{2\sqrt{\pi}} x^{-1/4} e^{-\frac{2}{3}x^{3/2}} \left(1 + \mathcal{O}(x^{-3/4})\right)$$

as $x \to +\infty$ *and* $\arg x = 0$;

$$y_{HM}(x) = \sqrt{-x/2}\left(1 + \mathcal{O}((-x)^{-3/2})\right) + c_-(-x)^{-1/4} e^{-\frac{2\sqrt{2}}{3}(-x)^{3/2}} \left(1 + \mathcal{O}(x^{-1/4})\right)$$

as $x \to \infty$ *and* $\arg x \in [\frac{2\pi}{3}, \frac{4\pi}{3})$, *where* $c_- = \frac{i2^{-7/4}}{\sqrt{\pi}}$.

Analysis of y_{HM} in the region Ω_0

- Change of variables:

$$t = \frac{2}{3}\sqrt{2}(-x)^{3/2}, \quad y(x) = \frac{\sqrt[3]{3t}}{2}h(t),$$

hence,

$$y_{HM} \mapsto h_{HM}, \quad \Omega_0 \mapsto \Omega_1 := \left\{ t \in \mathbb{C} \;\middle|\; |t| \geqslant 3, -\pi/2 \leqslant \arg t \leqslant 0 \right\}$$

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- We have $h_{HM}(t) = h_p(t) + h_e(t)$, where

$$h_p(t) = 1 - \frac{1}{9t^2} + \frac{h_2(t)}{t^4} \sim 1 - \frac{1}{9t^2}, \quad |h_2(t)| \leqslant \frac{6}{5},$$

$$h_e(t) = \frac{\widetilde{c}e^{-t}}{\sqrt{t}}(h_a(t) + \delta_1(t)) \sim \frac{\sqrt{2}\widetilde{c}e^{-t}}{\sqrt{t}},$$

with h_a being a quasi-solution and $|\delta_1(t)| \leqslant \frac{5}{2|t|^2}$

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- When $|x|$ becomes small, no asymptotic expansion can provide sufficient information about y_{HM} and the initial values at a finite point are needed

Analysis of y_{HM} in the finite region Ω_2

- When $|x|$ becomes small, no asymptotic expansion can provide sufficient information about y_{HM} and the initial values at a finite point are needed
- To get approximations of initial values at the origin with controlled error bound
  - Analysis of y_{HM} for $x \geqslant 3$:

$$\text{quasi-solution} \rightarrow \text{aproximations of } y_{HM}(3) \text{ and } y'_{HM}(3)$$

  - Analysis of y_{HM} for $0 \leq x \leq 3$:

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Analysis of y_{HM} in the finite region Ω_2

- From approximations of initial values at the origin, we obtain

$$|y_{HM}(x) - y_b(x)| < 6/5, \qquad x \in \Omega_2,$$

with y_b being the quasi-solution (a polynomial of degree 15)

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- Technical parts:
  - Contractive map in a suitable Banach space
  - Taylor series / fitting numerical data
  - Estimating a real/complex polynomial over an interval/ a domain

Part II – Gap probability at the hard edge for random matrix ensembles with pole singularities in the potential

The model

- A probability measure

$$\frac{1}{Z_n}(\det M)^{\alpha} \exp[-n \mathrm{tr}\, V_k(M)]\, \mathrm{d}M, \qquad \alpha > -1,$$

defined on the space of $n \times n$ positive definite Hermitian matrices where
  - Z_n: a normalization constant
  - $\mathrm{d}M$: flat complex Lebesgue measures on the entries
  - the potential

$$V_k(x) := V(x) + \left(\frac{t}{x}\right)^k, \qquad x \in (0, \infty), \quad t > 0$$

Eigenvalue distribution

- Joint probability density function of the eigenvalue distribution:

$$\frac{1}{C_n}\prod_{1\leq i<j\leq n}(x_j-x_i)^2\prod_{j=1}^{n}w(x_j),$$

where

$$w(x)=x^{\alpha}e^{-nV_k(x)}$$

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- Correlation kernel

$$K_n(x, y; t) = h_{n-1}^{-1} \sqrt{w(x)w(y)} \frac{\pi_n(x)\pi_{n-1}(y) - \pi_{n-1}(x)\pi_n(y)}{x - y},$$

where

$$\int_0^{\infty} \pi_j(x)\pi_m(x)w(x)\,\mathrm{d}x = h_j \delta_{j,m}$$

Motivations

- Statistics for zeta zeros and eigenvalues – probability distribution of Tuck's function

  [Berry-Shukla, '08]

- Quantum transport and electrical characteristics of chaotic cavities – eigenvalues of Wigner-Smith time-delay matrix

  [Brouwer-Frahm-Beenakker, '97&'99; Grabsch–Texier, '14]

- The field of spin-glasses – random matrix model arising in mean-field glassy systems

  [Akemann-Villamaina-Vivo,'14]

Recent progresses

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- Singularly perturbed GUE: $V_k(x) = \frac{1}{2}x^2 + \frac{t}{2x^2}, \quad \alpha = 0$
  - Double scaling limit of the partition function – related to the Painlevé III equation
    [Mezzadri-Mo, '09; Brightmore-Mezzadri-Mo,'15]
- Singularly perturbed LUE: $V_k(x) = x + \frac{t}{x}$
  - Connection with the Painlevé III equation for finite n
    [Chen-Its,'10]
  - Double scaling limit for the correlation kernel – model RH problem associated to the Painlevé III equation
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  - Double scaling limit for the correlation kernel – model RH problem associated to the Painlevé III equation

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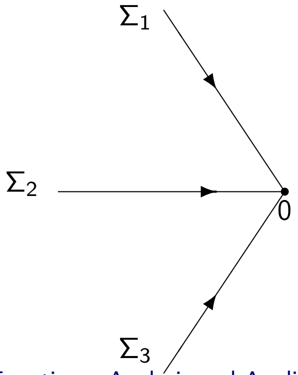
- General potential: $V_k(x) = V(x) + \left(\frac{t}{x}\right)^k$
  - Connection with a Painlevé III hierarchy

    [Atkin-Claeys-Mezzadri, '16]

A Riemann-Hilbert (RH) problem

(1) $\Psi(z;\lambda)$ is analytic in $\mathbb{C} \setminus \cup_{j=1}^{3}\Sigma_j$.

(2) Jump conditions:



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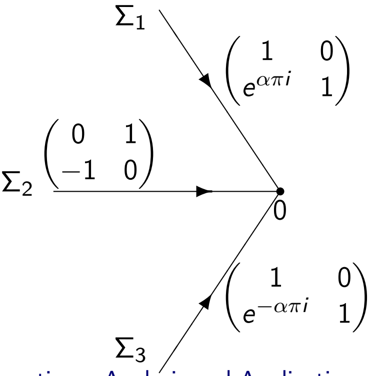

RH problem for Ψ

(3) As $z \to \infty$,

$$\Psi(z;\lambda) = \left(I + \frac{\begin{pmatrix} q(\lambda) & -ir(\lambda) \\ ip(\lambda) & -q(\lambda) \end{pmatrix}}{z} + \mathcal{O}(z^{-2}) \right) \times z^{-\frac{1}{4}\sigma_3} \frac{I + i\sigma_1}{\sqrt{2}} e^{\sqrt{z}\sigma_3}$$

(4) As $z \to 0$,

$$\Psi(z;\lambda) = \Psi_0(\lambda)(I + \mathcal{O}(z)) e^{-\left(-\frac{\lambda}{z}\right)^k \sigma_3} z^{\frac{\alpha}{2}\sigma_3} H_j$$

Remarks about the RH problem

- This RH problem is uniquely solvable for $k \in \mathbb{N}$, $\alpha > -1$ and $\lambda > 0$

  [Xu-Dai-Zhao,'14; Atkin-Claeys-Mezzadri,'16]

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- $\lambda = 0$: explicitly solvable in terms of modified Bessel functions
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- $\lambda = 0$: explicitly solvable in terms of modified Bessel functions

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- Connection with a Painlevé III hierarchy: $k+1$ ODEs for $k+1$ unknown functions $(\rho(\lambda), \ell_1(\lambda), \ldots, \ell_k(\lambda))$

$$\begin{cases} \rho = -\frac{1}{4\ell_k^2}((\ell_k^2)'' - 3(\ell_k')^2 + \tau_0), & p = 0, \\ \sum_{q=0}^{p} (\ell_{k-p+q+1}\ell_{k-q} - (\ell_{k-p+q}\ell_{k-q})'' + \\ \qquad 3\ell_{k-p+q}'\ell_{k-q}' - 4\rho\ell_{k-p+q}\ell_{k-q}) = \tau_p, & 1 \le p \le k \end{cases}$$

Remarks about the RH problem

Proposition (Atkin-Claeys-Mezzadri,'16)

Let

$$y_\alpha(\lambda) := -2i\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(r(\lambda^2)\right).$$

Then, $y_\alpha(\lambda)$ is a solution of the equation for ℓ_1 of the k-th member of the aforementioned Painlevé III hierarchy with

$$\tau_p = \begin{cases} 4^{2k+1}k^2, & p = 0, \\ -(-4)^{k+1}\alpha k, & p = k, \\ 0, & 0 < p < k \end{cases}$$

In addition, the asymptotics of r, hence y_α is known.

Double scaling limit of K_n at the hard edge

- $\left(\psi_1(z;\lambda), \psi_2(z;\lambda)\right)^t$: analytic extension of first column of $\Psi(z;\lambda)$ in the region bounded by Γ_1 and Γ_3

Double scaling limit of K_n at the hard edge

- $\left(\psi_1(z;\lambda),\psi_2(z;\lambda)\right)^t$: analytic extension of first column of $\Psi(z;\lambda)$ in the region bounded by Γ_1 and Γ_3
- If $t\to 0$ and $n\to\infty$ in such a way that $2^{-\frac{1}{k}}c_1 n^{\frac{2k+1}{k}}t\to\lambda>0$, then

$$\lim_{n\to\infty}\frac{1}{c_1 n^2}K_n\left(\frac{u}{c_1 n^2},\frac{v}{c_1 n^2};t\right)=K_{\mathrm{PIII}}(u,v;\lambda),$$

where

$$K_{\mathrm{PIII}}(u,v;\lambda):=e^{\alpha\pi i}\frac{\psi_1(-v;\lambda)\psi_2(-u;\lambda)-\psi_1(-u;\lambda)\psi_2(-v;\lambda)}{2\pi i(u-v)}$$

[Xu-Dai-Zhao,'14; Atkin-Claeys-Mezzadri,'16]

Main result

Theorem (Dai-Xu-LZ, '17)

Let $\mathcal{K}_{PIII}$ be the integral operator with kernel $K_{PIII}(u,v)\chi_{[0,s]}(v)$ acting on the function space $L^2((0,\infty))$. Then,

$$\ln\det(I-\mathcal{K}_{PIII}) = -\frac{s}{4} + \alpha s^{1/2} - \frac{\alpha^2}{4}\ln s$$
$$+\int_0^\lambda \frac{1}{2t}\left(r(t)+\frac{\alpha^2}{2}-\frac{1}{8}\right)\mathrm{d}t + \tau_\alpha + \mathcal{O}(s^{-1/2}), \qquad s\to+\infty,$$

where r is related to a Painlevé III hierarchy, $\tau_\alpha = \ln\left(\frac{G(1+\alpha)}{(2\pi)^{\alpha/2}}\right)$ with $G(z)$ being the Barnes G-function

About the proof

- Relies on the integrable (in the sense of IIKS) structure of K_{PIII} and a Deift/Zhou steepest analysis of the associated RH problem

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- Relies on the integrable (in the sense of IIKS) structure of K_{PIII} and a Deift/Zhou steepest analysis of the associated RH problem
- Large s asymptotics of Fredholm determinant associated with other Painlevé kernels:
  - Painlevé I hierarchy
    [Claeys-Its-Krasovsky,'10]
  - Painlevé II kernel (Hastings-McLeod solution)
    [Bothner-Its,'14]
  - Painlevé II kernel (Ablowitz-Segur solution)
    [Bothner-Buckingham,'17]
  - Painlevé XXXIV kernel
    [Xu-Dai,'17]

About the proof

Step 1 Large s asymptotics of $\frac{\mathrm{d}}{\mathrm{d}s}F(s;\lambda)$ with
$F(s;\lambda) := \ln\det(I - K_{\mathrm{PIII}})$

- ▷ Fact

$$\frac{\mathrm{d}}{\mathrm{d}s}\ln F(s;\lambda) = -\frac{e^{\alpha\pi i}}{2\pi i}\lim_{z\to -s}\left(X^{-1}(z)X'(z)\right)_{21}$$

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Step 2 Large s asymptotics of $\frac{\mathrm{d}}{\mathrm{d}\lambda}F(s;\lambda)$

- ▷ Fact

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}F(\lambda^2 s;\lambda^2) = r(\lambda^2)/\lambda - (\widehat{X}_\infty)_{12}$$

About the proof

Step 3 The constant term

- Fact

$$K_{\mathrm{PIII}}(u,v;\lambda)=K_{\mathrm{Bes}}(u,v)+\mathcal{O}(\lambda),\qquad \lambda\to 0,$$

where

$$K_{\mathrm{Bes}}(x,y)=\frac{J_\alpha(\sqrt{x})\sqrt{y}J_\alpha'(\sqrt{y})-\sqrt{x}J_\alpha'(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)}$$

About the proof

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$$K_{\mathrm{Bes}}(x,y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)}$$

- ▷ As $s \to +\infty$,

$$\ln\det(I - \mathcal{K}_{\mathrm{Bes}}) = -\frac{1}{4}s + \alpha s^{1/2} - \frac{\alpha^2}{4}\ln s + \ln\left(\frac{G(1+\alpha)}{(2\pi)^{\alpha/2}}\right) + O(s^{-1/2})$$

[Deift-Krasovsky-Vasilevska,'11]

Future work

- Tracy-Widom type formula for the gap probability?
  - Coupled Painlevé III system for $k = 1$

Thanks for your attention!