The Density Matrix for the Ground State of 1-d Impenetrable Bosons in a Harmonic Trap

N.S. Witte

Institute of Fundamental Sciences
Massey University
New Zealand

29 August 2017

A. A. Kapaev Memorial Workshop
Michigan Center for Applied Interdisciplinary Mathematics
August 25-29, 2017

- 1-d Impenetrable Bosons

The Plan

- 1-d Impenetrable Bosons
- Density Matrix and Bose Condensation

The Plan

- 1-d Impenetrable Bosons
- Density Matrix and Bose Condensation
- Homogeneous and Periodic Case

The Plan

- 1-d Impenetrable Bosons
- Density Matrix and Bose Condensation
- Homogeneous and Periodic Case
- Recurrence Relations

The Plan

- 1-d Impenetrable Bosons
- Density Matrix and Bose Condensation
- Homogeneous and Periodic Case
- Recurrence Relations
- Orthogonal Polynomial System & Semi-Classical OPS Class

- 1-d Impenetrable Bosons
- Density Matrix and Bose Condensation
- Homogeneous and Periodic Case
- Recurrence Relations
- Orthogonal Polynomial System & Semi-Classical OPS Class
- Lax pairs

- 1-d Impenetrable Bosons
- Density Matrix and Bose Condensation
- Homogeneous and Periodic Case
- Recurrence Relations
- Orthogonal Polynomial System & Semi-Classical OPS Class
- Lax pairs
- $L(1,1,3;2)$ member of degenerations from 2-variable Garnier system

Hamiltonian N bosons $x_j \in (-\infty, \infty)$

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{j=1}^{N} x_j^2 + 2c \sum_{1\le i<j\le N} \delta(x_i - x_j), \quad c > 0$$

Ground State eigenfunction $\Psi_0(x_1, \ldots, x_N)$ satisfies

$$H\Psi_0 = E_0\Psi_0$$

subject to boundary conditions

1. $\Psi_0|_{x_i=x_j+0} = \Psi_0|_{x_i=x_j-0}$
2. $\left(\frac{\partial\Psi_0}{\partial x_i} - \frac{\partial\Psi_0}{\partial x_j}\right)\Big|_{x_i=x_j+0} - \left(\frac{\partial\Psi_0}{\partial x_i} - \frac{\partial\Psi_0}{\partial x_j}\right)\Big|_{x_i=x_j-0} = 2c\ \Psi_0|_{x_i=x_j\pm 0}$

Bethe Ansatz solution without harmonic trap [Lieb+Liniger 1963]
Impenetrable Bose wavefunction $c \to \infty$ [Girardeau 1960].
The wave function for impenetrable bosons must vanish at coincident coordinates,

$$\Psi_0(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N) = 0 \text{ for } x_i = x_j, (i \neq j),$$

This means that for any fixed ordering of the particles $x_1 < x_2 < \ldots < x_N$ there is no distinction between impenetrable bosons and free fermions.

Schrödinger Operator

$$H_R = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{j=1}^{N} x_j^2$$

Normalised single particle eigenstates $\{\phi_k(x)\}_{k=0,1,...}$

$$\phi_k(x) = \frac{2^{-k}}{c_k} e^{-x^2/2} H_k(x), \quad (c_k)^2 = \pi^{1/2} 2^{-k} k!$$

$H_k(x)$ denotes the Hermite polynomial of degree k.
Ground state wave function Ψ_0 as a Slater determinant of distinct single-particle states

$$\Psi_0(x_1, \ldots, x_N) = \frac{1}{(N!)^{1/2}} \left| \det[\phi_k(x_j)]_{\substack{j=1,...,N \\ k=0,...,N-1}} \right|$$

where the factor of $(N!)^{-1/2}$ is included so that

$$\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_N \, |\Psi_0(x_1, \ldots, x_N)|^2 = 1.$$

Density Matrix is defined as

$$\rho_N(x;y) \equiv N \int_{\mathbb{R}} dx_2 \ldots \int_{\mathbb{R}} dx_N \Psi_0(x, x_2, \ldots, x_N)\Psi_0(y, x_2, \ldots, x_N),$$

so that

$$\int_{\mathbb{R}} dx\, \rho_N(x;x) = N$$

Using this one must solve the eigenvalue problem

$$\int_{\mathbb{R}} \rho_N(x;y)\psi_k(y)\, dy = \lambda_k \psi_k(x), \quad k \in \mathbb{Z}_{\geq 0}.$$

Note

$$\lambda_0 > \lambda_1 > \ldots > \lambda_{N-1} > 0, \quad \sum_{j=0}^{N-1} \lambda_j = N$$

Recall

$$\rho_N(x;y) = N \int_{\mathbb{R}} dx_2 \dots \int_{\mathbb{R}} dx_N \Psi_0(x, x_2, \dots, x_N)\Psi_0(y, x_2, \dots, x_N)$$

Vandermonde determinant formula with $p_j(x)$ a monic polynomial of degree j,

$$\det[p_{j-1}(x_k)]_{j,k=1,\dots,N} = \det[x_k^{j-1}]_{j,k=1,\dots,N} = \prod_{1\leq j<k\leq N}(x_k - x_j)$$

Product formula for Ψ_0

$$\Psi_0(x_1, \dots, x_N) = \frac{1}{C_N}\prod_{j=1}^{N} e^{-x_j^2/2} \prod_{1\leq j<k\leq N} |x_k - x_j|, \quad (C_N)^2 = N!\prod_{l=0}^{N-1}(c_l)^2.$$

This coincides precisely with the multivariate probability distribution function (p.d.f.) for a particular class of random matrices

$$|\Psi_0|^2 = \mathbb{E}_{\mathrm{GUE}_N}$$

namely, GUE_N, the Gaussian unitary ensemble of $N \times N$ complex Hermitian matrices, Density matrix $\rho_{N+1}(x;y)$ can be written as an average over this eigenvalue p.d.f

$$\rho_{N+1}(x;y) = \frac{1}{c_N^2} e^{-x^2/2-y^2/2}\mathbb{E}_{\mathrm{GUE}_N}\left(\prod_{l=1}^{N}|x - x_l||y - x_l|\right)$$

Criteria for Bose-Einstein condensation (BEC) are -

- Onsager-Penrose 1956 criteria for BEC

$$\frac{\lambda_0(N)}{N} = \begin{cases} e^{\mathrm{O}(1)} & \text{BEC} \\ \mathrm{o}(1) & \text{no BEC} \end{cases} \quad \text{as} \quad N \to \infty$$

- Off Diagonal Long Range Order criteria of Yang 1962

$$\rho_\infty(x) \underset{|x|\to\infty}{\rightarrow} C > 0$$

NB: already in thermodynamic limit $N \to \infty$

[Jimbo Miwa Mori Sato 1980] Density Matrix $\rho_\infty(x;0)$ is now identified with a τ-function of the fifth Painlevé system

$$\rho_\infty(x;0) = \rho_0 \exp\Big(\int_0^{\pi\rho_0 x} \sigma_V(t)\, \frac{dt}{t} \Big),$$

where ρ_0 denotes the bulk density. σ_V satisfies the non-linear equation

$$(x\sigma_V'')^2 + 4(x\sigma_V' - \sigma_V - 1)\Big(x\sigma_V' - \sigma_V + (\sigma_V')^2\Big) = 0$$

subject to the $x \to 0$ boundary condition

$$\sigma_V(x) \underset{x\to 0}{\sim} -\frac{x^2}{3} + \frac{x^3}{3\pi} + \mathrm{O}(x^4).$$

[Lenard 1972] Eigenvalues of the density matrix are Fourier coefficients

$$\lambda_k = \int_0^L \rho_N(x;0) e^{2\pi i k x/L}\, dx.$$

The $N \to \infty$ asymptotic expansion

$$\rho_N(x;0) \sim \rho_0 A \left(\frac{\pi}{N \sin(\pi\rho_0 x/N)} \right)^{1/2}, \quad A = \frac{G^4(3/2)}{\sqrt{2\pi}}$$

where $G(x)$ denotes the Barnes' G-function, valid for x/N fixed. Therefore

$$\lambda_0 \sim c\sqrt{N}$$

a specific c computable from the above.

[FFGW 2003] Finite system of $N+1$ particles on a ring of circumference L, $x \in [0, L]$.
Wavefunction

$$\Psi_0(x_1, \ldots, x_N) = \frac{1}{L^{N/2}(N!)^{1/2}} \prod_{1 \le j < k \le N} 2 \left| \sin \frac{\pi(x_k - x_j)}{L} \right|$$

Density Matrix $\rho_{N+1}(x;0)$

$$\rho_{N+1}(x;0) = \frac{1}{L} \mathbb{E}_{U(N)} \left(\prod_{l=1}^{N} \left| 2 \sin\left(\frac{\pi x}{L} - \frac{\theta_l}{2} \right) \right| \left| 2 \sin\left(\frac{\theta_l}{2} \right) \right| \right)$$

Density Matrix can be identified with a member of a τ-function sequence in the sixth Painlevé system with Okamoto parameters

$$\mathbf{b} = \left(\tfrac{1}{2}N, 1 + \tfrac{1}{2}N, \tfrac{1}{2}N, -1 - \tfrac{1}{2}N \right).$$

Thermodynamic limit made by substituting $x = e^{2it/N}$ and taking $N \to \infty$ with N/L fixed.

The general Heine identity

$$N!\,\det\left[\int_{\mathbb{R}} dx\; w(x)x^{j+k-2}\right]_{j,k=1,\ldots,N} = \int_{\mathbb{R}} dx_1 \cdots \int_{\mathbb{R}} dx_N \prod_{l=1}^{N} w(x_l) \prod_{1\le j<k\le N} |x_j - x_k|^2$$

Determinant formula

$$\rho_{N+1}(x;y) = \frac{1}{c_N^2} e^{-x^2/2-y^2/2} \det[a_{j,k}(x;y)]_{j,k=1,\ldots,N}$$

with matrix elements

$$a_{j,k}(x;y) = \frac{2^{-j-k+2}}{c_{j-1}c_{k-1}} \int_{-\infty}^{\infty} dt\; |x-t||y-t| H_{j-1}(t) H_{k-1}(t) e^{-t^2}$$

To simplify further, we note

$$|x-t||y-t| = \begin{cases} (x-t)(y-t), & t \notin [x,y] \\ -(x-t)(y-t), & t \in [x,y] \end{cases}$$

Orthogonal polynomial system $\{p_n(t)\}_{n=0}^{\infty}$ defined by the orthogonality relations

$$\int_{\mathbb{R}} dt\, w(t) p_n(t) t^m = \begin{cases} 0 & 0 \leq m < n \\ h_n & m = n \end{cases}$$

A consequence of the orthogonality relation is the three term recurrence relation

$$a_{n+1} p_{n+1}(t) = (t - b_n) p_n(t) - a_n p_{n-1}(t), \quad n \geq 1$$

Central objects in our probabilistic model are the *Hankel determinants*

$$\Delta_n := \det[\mu_{j+k-2}]_{j,k=1,\ldots,n}, n \geq 1, \quad \Delta_0 := 1,$$

in terms of the *moments* of the weight $\{\mu_n\}_{n=0,1,\ldots,\infty}$

$$\mu_n := \int_{\mathbb{R}} dt\, w(t) t^n$$

Alternatively

$$\Delta_n = \frac{1}{n!} \int_{\mathbb{R}} dt_1 \ldots \int_{\mathbb{R}} dt_n \prod_{l=1}^{n} w(t_l) \prod_{1 \leq j < k \leq n} (t_k - t_j)^2, \quad n \geq 1$$

result in orthogonal polynomial theory

$$a_n^2 = \frac{\Delta_{n+1} \Delta_{n-1}}{\Delta_n^2}, \quad n \geq 1,$$

[Bauldry 1990, Bonan and Clark 1990]

Orthogonal polynomials satisfy a system of coupled first order linear differential equations with respect to t ($' \equiv d/dt$)

$$Wp_n' = (\Omega_n - V)p_n - a_n\Theta_n p_{n-1}, \quad n \geq 1,$$

with the coefficient functions $V(t), W(t), \Theta_n(t), \Omega_n(t)$.

Compatibility Condition → Coefficient functions satisfy the recurrence relations

$$\Omega_{n+1} + \Omega_n = (t - b_n)\Theta_n, \quad n \geq 0,$$
$$(\Omega_{n+1} - \Omega_n)(t - b_n) = W + a_{n+1}^2\Theta_{n+1} - a_n^2\Theta_{n-1}, \quad n \geq 0$$

[Maroni, ... 1985 ->]

Semi-Classical weights defined by

$$\frac{1}{w(t)}\frac{d}{dt}w(t) = \frac{2V(t)}{W(t)}$$

with $V(t)$ and $W(t)$ irreducible polynomials in t.

Deformed Hermite Orthogonal Polynomials with weight

$$\begin{aligned} w(t;x,y) &= |t-x||t-y|e^{-t^2}, \quad t \in \mathbb{R}, \\ &= [1-2\chi_{(x,y)}(t)](t-x)(t-y)e^{-t^2} \end{aligned}$$

where

$$\chi_I(t) = \begin{cases} 1 & \text{if } t \in I \\ 0 & \text{otherwise} \end{cases}$$

[Magnus 1995, ...]
Numerator and denominator polynomials

$$W(t) = (t-x)(t-y), \quad 2V(t) = -2t^3 + 2(x+y)t^2 + 2t(1-xy) - x - y$$

N.B. we have finite singularities at $t = x, y$.
Coefficient functions are

$$\Theta_n(t) = -2t^2 + 2(x+y-b_n)t \\ + 2n + 1 + 2(x+y)b_n + 2(1-xy) - 2[a_{n+1}^2 + a_n^2 + b_n^2], \quad n \geq 0$$

and

$$\Omega_n(t) = -t^3 + (x+y)t^2 + (n+1-xy-2a_n^2)t \\ + \sum_{i=0}^{n-1} b_i + (x+y)(2a_n^2 - n - \frac{1}{2}) - 2a_n^2(b_n + b_{n-1}), \quad n \geq 1$$

Proposition

Density matrix is given by

$$\rho_{N+1}(x;y) = \frac{(N+1)!}{C_{N+1}^2} e^{-x^2/2-y^2/2} \Delta_N,$$

and

$$\Delta_{N+1} = a_N^2 \frac{\Delta_N^2}{\Delta_{N-1}},$$

subject to $\Delta_0 = 1, \Delta_1 = \mu_0$. *Coupled recurrences for the polynomial coefficients are*

$$\sum_{i=0}^{n} b_i + (n+1-xy)b_n - b_n^3 + (x+y)[b_n^2 - n - 1] \\ + a_{n+1}^2[x+y-2b_n-b_{n+1}] - a_n^2[x+y-2b_n-b_{n-1}] = 0$$

and

$$a_{n+1}^2[2n+5-2xy+2(x+y)(b_{n+1}+b_n) - 2(a_{n+2}^2+a_{n+1}^2+b_{n+1}^2+b_n^2+b_{n+1}b_n)] \\ - a_n^2[2n+1-2xy+2(x+y)(b_n+b_{n-1}) - 2(a_n^2+a_{n-1}^2+b_n^2+b_{n-1}^2+b_nb_{n-1})] \\ + (b_n-x)(b_n-y) = 0$$

The initial data for b_n and a_n^2 are given by

$$b_{-1} = 0, \quad b_0 = \frac{\mu_1}{\mu_0}, \quad a_0^2 = 0, \quad a_1^2 = \frac{\mu_0\mu_2 - \mu_1^2}{\mu_0^2}$$

where for $x < y$

$$\mu_0 = \sqrt{\pi}(\frac{1}{2} + xy)[1 + erf(x) - erf(y)] + ye^{-x^2} - xe^{-y^2}$$
$$\mu_1 = -\frac{1}{2}\sqrt{\pi}(x + y)[1 + erf(x) - erf(y)] - e^{-x^2} + e^{-y^2}$$
$$\mu_2 = \sqrt{\pi}(\frac{3}{4} + \frac{1}{2}xy)[1 + erf(x) - erf(y)] + (y - \frac{1}{2}x)e^{-x^2} - (x - \frac{1}{2}y)e^{-y^2}$$

Deformed Hermite Orthogonal Polynomials with weight

$$w(t;x,y) = (t-x)^{\mu}(t-y)^{\nu}e^{-t^2}, \quad t \in \Gamma$$

where $\partial\Gamma \in \{-\infty, x, y, \infty\}$.
Numerator and denominator polynomials

$$W(t) = (t-x)(t-y), \quad 2V(t) = -2t(t-x)(t-y) + \nu(t-x) + \mu(t-y)$$

Coefficient functions are

$$\Theta_n(t) = -2(t-x)(t-y) - 2b_n(t-x-y) + \theta_n, \quad n \geq 0$$

and

$$\Omega_n(t) = -t(t-x)(t-y) - (2a_n^2 - n)(t-x-y) + \frac{1}{2}\nu(t-x) + \frac{1}{2}\mu(t-y) + \omega_n, \quad n \geq 1$$

Proposition

Coupled first order recurrences $n \mapsto n+1$ *for system* $\{a_n^2, b_n, \theta_n, \omega_n\}$

$$\theta_n + 2b_n x = \frac{[\omega_n + (2a_n^2 - n)x][\omega_n + (2a_n^2 - n)x - \nu(x-y)]}{a_n^2(\theta_{n-1} + 2b_{n-1}x)}$$

$$\theta_n + 2b_n y = \frac{[\omega_n + (2a_n^2 - n)y][\omega_n + (2a_n^2 - n)y + \mu(x-y)]}{a_n^2(\theta_{n-1} + 2b_{n-1}y)}$$

$$a_{n+1}^2 + a_n^2 = -b_n^2 + \frac{1}{2}(2n + 1 + \mu + \nu - \theta_n)$$

$$\omega_{n+1} + \omega_n = (x+y)\theta_n + b_n(2xy - \theta_n) - \mu x - \nu y$$

2×2 matrix system

$$Y_n(t;x,y) = \begin{pmatrix} p_n & \epsilon_n/w \\ p_{n-1} & \epsilon_{n-1}/w \end{pmatrix}$$

where the *associated functions* are

$$\epsilon_n(s;x,y) = \int_\Gamma dt\; w(t)\frac{p_n(t)}{s-t}, \quad s \notin \Gamma$$

Proposition

Assume $\mathrm{Re}\mu > 0, \mathrm{Re}\nu > 0$ *and* $x \neq y$*. The spectral derivatives satisfy*

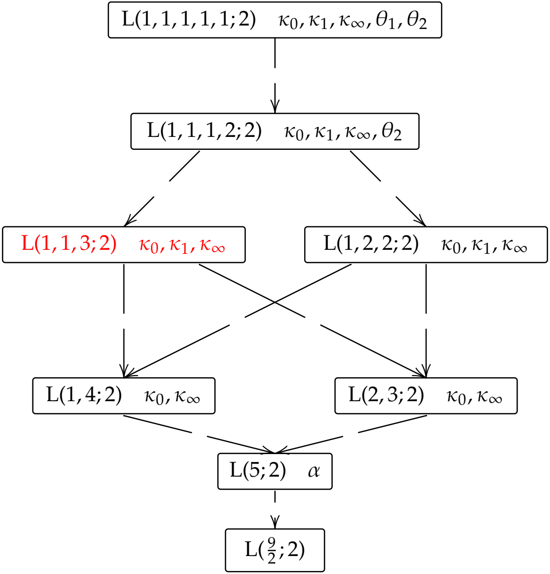
$$\frac{\partial}{\partial t} Y_n = \left\{ 2t\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + 2a_n\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{A_x}{t-x} + \frac{A_y}{t-y} \right\} Y_n \equiv AY_n$$

where the residue matrices are

$$A_x(t;x,y;\mu,\nu) \equiv \begin{pmatrix} \dfrac{\omega_n + (2a_n^2 - n)y}{x-y} & -\dfrac{a_n(\theta_n + 2b_n y)}{x-y} \\ \dfrac{a_n(\theta_{n-1} + 2b_{n-1}y)}{x-y} & -\dfrac{\omega_n + (2a_n^2 - n)y}{x-y} - \mu \end{pmatrix}, \qquad (1)$$

and $A_y = A_x|_{x\leftrightarrow y, \mu\leftrightarrow\nu}$.

Properties: $\mathrm{Tr}A_x = -\mu$, $\det A_x = 0$



$L(1,1,1,1,1;2)$ $\kappa_0, \kappa_1, \kappa_\infty, \theta_1, \theta_2$
$L(1,1,1,2;2)$ $\kappa_0, \kappa_1, \kappa_\infty, \theta_2$
$L(1,1,3;2)$ $\kappa_0, \kappa_1, \kappa_\infty$
$L(1,2,2;2)$ $\kappa_0, \kappa_1, \kappa_\infty$
$L(1,4;2)$ κ_0, κ_∞
$L(2,3;2)$ κ_0, κ_∞
$L(5;2)$ α
$L(\frac{9}{2};2)$

Proposition

Assume $\mathrm{Re}\mu > 0, \mathrm{Re}\nu > 0$ *and* $x \neq y$. *The deformation derivatives with respect to* x *or* y *satisfy*

$$\frac{\partial}{\partial x} Y_n = \left\{ \frac{1}{2} \frac{1}{x-y} \begin{pmatrix} -\theta_n - 2b_n y & 0 \\ 0 & \theta_{n-1} + 2b_{n-1} y \end{pmatrix} - \frac{A_x}{t-x} \right\} Y_n \equiv B_x Y_n$$

$$\frac{\partial}{\partial y} Y_n = \left\{ \frac{1}{2} \frac{1}{x-y} \begin{pmatrix} \theta_n + 2b_n x & 0 \\ 0 & -\theta_{n-1} - 2b_{n-1} x \end{pmatrix} - \frac{A_y}{t-y} \right\} Y_n \equiv B_y Y_n$$

Special diagonal case: $x = y = 0$ yields

$$b_n = 0$$

and

$$a_{n+1}^2 \left[2n + 5 - 2a_{n+2}^2 - 2a_{n+1}^2 \right] = a_n^2 \left[2n + 1 - 2a_n^2 - 2a_{n-1}^2 \right]$$

with

$$a_0^2 = 0, \quad a_1^2 = \frac{3}{2}$$

Solution?

Special diagonal case: $x = y = 0$ yields

$$b_n = 0$$

and

$$a_{n+1}^2 \left[2n + 5 - 2a_{n+2}^2 - 2a_{n+1}^2 \right] = a_n^2 \left[2n + 1 - 2a_n^2 - 2a_{n-1}^2 \right]$$

with

$$a_0^2 = 0, \quad a_1^2 = \frac{3}{2}$$

Solution?

$$a_n^2 = \frac{n+1}{2} - \frac{1}{2}(-1)^n$$

Special diagonal case: $x = y = 0$ yields

$$b_n = 0$$

and

$$a_{n+1}^2 \left[2n + 5 - 2a_{n+2}^2 - 2a_{n+1}^2 \right] = a_n^2 \left[2n + 1 - 2a_n^2 - 2a_{n-1}^2 \right]$$

with

$$a_0^2 = 0, \quad a_1^2 = \frac{3}{2}$$

Solution?

$$a_n^2 = \frac{n+1}{2} - \frac{1}{2}(-1)^n$$

or for general $y = x \neq 0$

Special diagonal case: $x = y = 0$ yields

$$b_n = 0$$

and

$$a_{n+1}^2 \left[2n + 5 - 2a_{n+2}^2 - 2a_{n+1}^2 \right] = a_n^2 \left[2n + 1 - 2a_n^2 - 2a_{n-1}^2 \right]$$

with

$$a_0^2 = 0, \quad a_1^2 = \frac{3}{2}$$

Solution?

$$a_n^2 = \frac{n+1}{2} - \frac{1}{2}(-1)^n$$

or for general $y = x \neq 0$

$$a_n^2 = \frac{n}{2} \frac{\left[(n+2)H_{n+1}^2 - (n+1)H_{n+2}H_n \right] \left[n\, H_{n-1}^2 - (n-1)H_n H_{n-2} \right]}{\left[(n+1)H_n^2 - n\, H_{n+1}H_{n-1} \right]^2}$$

Where to from here?

- ➩ Can the density matrix eigenvalues (or just the lowest one) be accessed directly and analytically using these ideas?
- ➩ the theory is easily generalisable to calculate other quantities, higher correlations etc,
- ➩ take the thermodynamic limit,
- ➩ finite temperature extensions may be possible within the structures,
- ➩ the situation for finite interparticle repulsion $c < \infty$ is more challenging,
- ➩ certainly other non-integrable potentials can be handled with a little more effort with the resulting difference and differential systems being non-integrable
- ➩ extensions to 2 dimensions would require a major task at the level of the basic mathematical theory

- M. D. Girardeau. *Relationship between systems of impenetrable bosons and fermions in one dimension.* J. Math. Phys., **1**:516-523, 1960.
- E. H. Lieb and W. Liniger. *Exact analysis of an interacting Bose gas. I. The general solution and the ground state.* Phys. Rev. (2), **130**:1605-1616, 1963.
- A. Lenard. *Momentum distribution in the ground state of the one-dimensional system of impenetrable bosons.* J. Math. Phys., **5**(7):930-943, 1964.
- H. G. Vaidya and C. A. Tracy. *One particle reduced density matrix of impenetrable bosons in one dimension at zero temperature.* J. Math. Phys., **20**(11):2291-2312, 1979.
- M. Jimbo, T. Miwa, Y. Môri, and M. Sato. *Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent.* Phys. D, **1**(1):80-158, 1980.
- H. Kimura. *The degeneration of the two-dimensional Garnier system and the polynomial Hamiltonian structure.* Ann. Mat. Pura Appl. **155**:(4), 25-74, 1989.
- A. P. Magnus. *Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials*. J. Comput. Appl. Math., **57**(1-2):215-237, 1995.
- P. J. Forrester, N. E. Frankel, T. M. Garoni, and N. S. Witte. *Painlevé transcendent evaluations of finite system density matrices for 1d impenetrable bosons.* Comm. Math. Phys., **238**(1-2):257-285, 2003.
- H. Kawamuko. *On the Garnier system of half-integer type in two variables.* Funkcial. Ekvac., **52**(2):181-201, 2009.