One dimensional free fermions at finite temperature and the MNS matrix model

joint work with Karl Liechty

Dong Wang

National University of Singapore

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Moshe–Neuberger–Shapiro matrix model

Let M be an $n \times n$ random Hermitian matrix, with density

$$\begin{aligned} P(M)dM &\propto \int dU \left[e^{-\frac{1}{2}\frac{1-q}{1+q}\operatorname{Tr} M^2} e^{-\frac{1}{2}\frac{q}{1-q^2}\operatorname{Tr}([U,M][U,M]^\dagger)} \right] dM \\ &\propto e^{-\frac{1}{2}\frac{1+q^2}{1-q^2}\operatorname{Tr} M^2} \left[\int dU e^{\frac{q}{1-q^2}\operatorname{Tr}(UMU^{-1}M)} \right] dM. \end{aligned}$$

where $q \in [0, 1)$, and the integration of U is over the unitary group with the Haar measure, and $[U, M] = UM - MU$.

Eigenvalue distribution

Using the HCIZ formula, we have that if $x_1, \ldots, x_n$ are eigenvalues of M, then

$$\int dU e^{\frac{q}{1-q^2} \operatorname{Tr}(UMU^{-1}M)} \propto \frac{\det(e^{\frac{q}{1-q^2} x_i x_j})_{i,j=1}^n}{\Delta(x)^2},$$

where $\Delta(x) = \prod_{1 \le i < j \le n}(x_i - x_j)$ is the Vandermonde determinant. Hence the distribution of the eigenvalues of M is

$$P(x_1, \ldots, x_n) \propto \det \left(e^{-\frac{1}{4}\frac{1+q^2}{1-q^2}(x_i^2+x_j^2)+\frac{q}{1-q^2}x_i x_j} \right)_{i,j=1}^n .$$

This model was proposed by the three authors (1994) to as a interpolating model from the Gaussian Unitary Ensemble ($q = 0$) to a random matrix whose eigenvalues are independent ($q \to 1$). So locally, it brings us a transition from Sine distribution to Poisson distribution, and from Tracy–Widom distribution to Gumbel distribution at the edge.

Relation to free fermions: $q = 0$ case

- The density function is

$$P(x_1, \ldots, x_n) = \frac{1}{n!} \left| \det(\varphi_{i-1}(x_j))_{i,j=1}^{n} \right|^2,$$

where (H_k is the (probabilists') Hermite polynomial)

$$\varphi_k(x) = \frac{1}{C_k} H_k(x) e^{-x^2/4}, \quad \int \varphi_k(x)^2 dx = 1.$$

- Consider the 1-dimensional harmonic oscillator in quantum mechanics. The energy levels are $1/2, 3/2, \ldots, (2k-1)/2, \ldots$, and the corresponding wave functions are $\varphi_0(x), \varphi_1(x), \ldots, \varphi_k(x), \ldots$.
- If we consider n independent particles, all in 1D harmonic oscillator movement, and suppose they are **fermions**, then their probability density function is $P(x_1, \ldots, x_n)$ above, providing that they are in the **ground state**.

Relation to free fermions: $q > 0$ case

- The density function, after some algebraic manipulation, becomes

$$P(x_1, \ldots, x_n) = \frac{1}{Z_n(q)} \sum_{0 \le k_1 < k_2 < \cdots < k_n} \begin{vmatrix} \varphi_{k_1}(x_1) & \ldots & \varphi_{k_1}(x_n) \\ \vdots & & \vdots \\ \varphi_{k_n}(x_1) & \ldots & \varphi_{k_n}(x_n) \end{vmatrix}^2 \times q^{k_1 + \cdots + k_n + n/2}.$$

- Each array $(k_1, k_2, \ldots, k_n)$ indexes an eigenstate for the free fermion system, with energy $k_1 + \cdots + k_n + n/2$. By the **Boltzmann distribution** of particles in an equilibrium system at **temperature** T, the probability that the free fermion system in state $(k_1, k_2, \ldots, k_n)$ is $\propto \exp(-(k_1 + \cdots + k_n + n/2)/(\kappa T))$ where κ is the Boltzmann constant (later assume to be 1).
- Let $q = e^{-\beta} = e^{-1/T}$, we have that the distribution of the eigenvalues of the MNS model is also the distribution of the particles in the free fermion system at temperature T.

Johansson's approach

- The difficulty of solving the MNS model is that it is NOT a determinantal process.
- Johansson (2007), inspired by Moshe, Neuberger and Shapiro, took an alternative approach. The free fermion model corresponding to the MNS matrix model is a **canonical ensemble**, that is, the total energy is random but the particle number is fixed. A fundamental law of statistical physics tells us that the canonical ensemble should be equivalent to the **grand canonical ensemble** physically, where in a grand canonical ensemble both the total energy and the particle number are random.
- Then why not to study the grand canonical ensemble instead, (although it does not have a matrix interpretation)?

Basic concepts about determinantal processes

If a point process on $\mathbb{R}$ is a **determinantal process**, then it has a **correlation kernel** $K(x, y)$, such that the k-correlation function of the point process is expressed by

$$R^{(k)}(x_1, \dots, x_k) = \det(K(x_i, x_j))_{i,j=1}^{k}.$$

Note that the correlation kernel is not unique: $f(x)K(x, y)f(y)^{-1}$ is a correlation kernel too. Also, if the number of point is fixed to be n, then the rank of the correlation kernel is n.
Another important quantity that can be simply expressed by the correlation kernel is the **gap probability**, that is, the probability that no point is in a certain subset $A \subseteq \mathbb{R}$.

$$P(x_1, \dots, x_n \in A) = \det(I - K(x, y)\chi_{A^c}),$$

where the right-hand side is the Fredholm determinant.

The grand canonical ensemble of free fermions

The grand canonical ensemble is realized as follows.

- In probability p_n $(n = 0, 1, 2, \dots)$, there are n particles, and under the condition that the particle number is n, the behaviour of the particles is the same as those in the n-particle canonical ensemble.
- Here we assume

$$p_n = \frac{1}{Z(q;\lambda)} \lambda^n Z_n(q).$$

  where $Z_n(q)$ is the normalization constant for n-particle canonical ensemble, and $Z(q;\lambda)$ is the overall normalization constant.
- Then the distribution of the particles is a determinantal process, that is, the k-correlation functions are expressed by $k \times k$ determinants of the correlation kernel. (Correlation functions are well defined when the total particle number is varying.)

The grand canonical ensemble of free fermions (continued)

- The correlation kernel for the grand canonical ensemble is

$$K_\lambda(x,y) = \sum_{k=0}^{\infty} \frac{\lambda q^{k+1/2}}{1+\lambda q^{k+1/2}} \varphi_k(x)\varphi_k(y)$$
$$= \sum_{k=0}^{\infty} \frac{1}{e^{\beta(\epsilon_k-\mu)}+1} \varphi_k(x)\varphi_k(y),$$

  where $\epsilon_k = k + 1/2$ is the k-th energy level, and $\mu = \beta^{-1}\log\lambda$ is the **chemical potential**. The factor $1/(e^{\beta(\epsilon_k-\mu)})$ is the **Fermi factor**.

- To make the grand canonical ensemble equivalent to the canonical ensemble with particle number n, we need

$$E(\text{particle number}) = \sum_{n=0}^{\infty} \frac{\lambda q^{k+1/2}}{1+\lambda q^{k+1/2}} \approx n,$$

  or approximately, $\lambda = q^{-n} - 1$.

Asymptotics for the grand canonical ensemble: Edge (from Tracy–Widom to Gumbel)

- Suppose $\beta = cn^{-1/3}$, or equivalently, $q = e^{-cn^{-1/3}}$, and the expectation of the particle number is n. Then the density of particles has the usual semicircle law, and the correlation kernel at $2\sqrt{n}$, the edge of the semicircle, has scale $n^{-1/6}$, and the limit

$$K_{\text{crossover}}(\xi, \eta) = \int_{-\infty}^{\infty} \frac{1}{e^{cr} + 1} \operatorname{Ai}(\xi - r) \operatorname{Ai}(\eta - r) dr.$$

- As $c \to +\infty$, it converges to the Airy kernel

$$K_{\text{TW}}(\xi, \eta) = \int_{-\infty}^{0} \operatorname{Ai}(\xi - r) \operatorname{Ai}(\eta - r) dr.$$

that leads to the Tracy–Widom distribution. As $c \to 0$, although not very obvious, this correlation kernel yields the Gumbel distribution that is common in the limit of extreme values.

Asymptotics for the grand canonical ensemble: Bulk (from semicircle to Gaussian)

- Suppose $\beta = cn^{-1}$, or equivalently, $q = e^{-cn^{-1}}$, and the expectation of the particle number is n. Then the density of particles at $2\sqrt{n}x$ is proportional to

$$-\frac{1}{\sqrt{\pi c}} \mathrm{Li}_{1/2}(-(e^c - 1)e^{-cx^2}),$$

  where $\mathrm{Li}_{1/2}$ is the polylogarithm function with parameter $1/2$, defined by the integral

$$\mathrm{Li}_{1/2}(-z) = -2\sqrt{\pi} \int_0^\infty \frac{1}{e^{t^2}/z + 1} dt.$$

  and $\mathrm{Li}_{1/2}(z) = \sum_{k=1}^\infty z^k/\sqrt{k}$ if $|z| < 1$.
- As $c \to \infty$, the density converges to the semicircle law supported on $[-1, 1]$, and as $c \to 0$, the density converges to e^{-cx^2} times a constant, which is the Gaussian distribution.

Figures for the limiting empirical density function

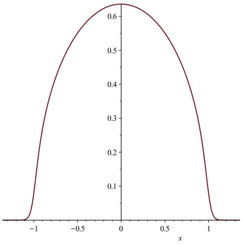


Figure: the density of particles when $c = 20$.

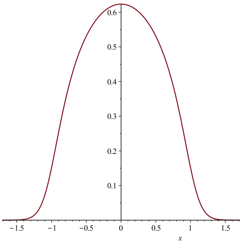


Figure: the density of particles when $c = 5$.

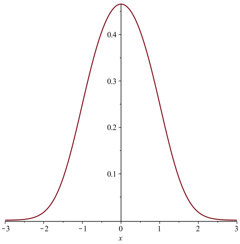


Figure: the density of particles when $c = 1$.

Asymptotics for the grand canonical ensemble: Bulk (from Sine to Poisson)

- Let the values of β, q and the expectation of the particle number be chosen as in last slide. Then the limiting correlation kernel at $2\sqrt{n}x$, scaled at order $n^{-1/2}$, is proportional to

$$\int_0^\infty \frac{1}{e^{t^2}/[(e^c-1)e^{-cx^2}]+1}\cos(t(\xi-\eta))dt.$$

- It is not hard to see that as $c \to \infty$, the kernel becomes the Sine kernel

$$\int_0^{\sqrt{c(1-x^2)}} \cos(t(\xi-\eta))dt = \frac{\sin(\sqrt{c(1-x^2)}(\xi-\eta))}{\sqrt{c(1-x^2)}(\xi-\eta)},$$

  and as $c \to 0$, it yields the Poisson distribution.
- The **non-universality** seems to be interesting.

Conanical ensemble and the MNS matrix model

The results presented above are not new. The three authors M–N–S defined the model and pointed out that the grand canonical ensemble is easier; all asymptotics for the grand canonical ensemble were essentially computed by Johansson, and recently physicists Dean, Le Dousal, Majumdar, and Schehr rekindled the interest on this model, and used an "almost" rigorous argument to assert that the same asymptotics hold for the canonical ensemble, that is, the MNS matrix model. Then what is left for one more paper on this topic?

Theorem

All the asymptotics obtained for the grand canonical ensemble are valid for the canonical ensemble/MNS matrix model too.

And our result is really rigorous.

Remark

Although the canonical ensemble/MNS matrix model is not determinantal, the local limits are determinantal processes, so the results are also expressed by correlation kernels.

Algebraic formulas

To state the exact formulas for the correlation functions and the gap probability for the canonical ensemble, we define the kernel

$$K(x, y; z) = \sum_{k=0}^{\infty} \frac{q^k z}{1 + q^k z} \varphi_k(x)\varphi_k(y),$$

where $x, y \in \mathbb{R}$, and $z \in \mathbb{C}$ is a parameter.

- Correlation functions

$$R^{(m)}(x_1, \dots, x_m) =$$
$$q^{-\binom{n}{2}}(q; q)_n \frac{1}{2\pi i} \oint_0 \frac{(-z; q)_\infty}{z^n} \det(K(x_i, x_j; z))_{i,j=1}^m \frac{dz}{z}.$$

- Gap probability

$$P(\text{all particles are in } A) =$$
$$q^{-\binom{n}{2}}(q; q)_n \frac{1}{2\pi i} \oint_0 \frac{(-z; q)_\infty}{z^n} \det(1 - K(x, y; z)\chi_{A^c}) \frac{dz}{z}.$$

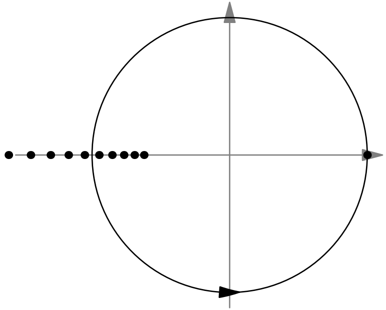
Here $(x; q)_n = (1 - x)(1 - qx)(1 - q^2 x) \cdots (1 - q^{n-1} x)$.

Idea of saddle point analysis

For the asymptotic analysis, we can let the contour for z be a circle centred at 0 with radius $q^{-n+1/2}$. Then find that $z = q^{-n+1/2}$ is the unique saddle point. So the contour integral is approximated by the correlation function/gap probability given by the correlation kernel

$$K(x, y; q^{-n+1/2}),$$

which is almost identical to the correlation for the grand canonical ensemble.



The tricky part is at $-q^{-n+1/2}$, where the contour goes between poles of $\det(1 - K(x, y; z)\chi_{A^c})$. (But they are not poles of the integrand.)

Proof of the algebraic identities

- Suppose f_n is a quantity for the n-particle canonical ensemble, (for example, gap probability, k-correlation function, linear statistics, etc). The grand canonical ensemble is the superposition of n-particle canonical ensembles with $n = 0, 1, 2, \dots$. If the chemical potential μ is higher, then the particle number is more likely to be smaller, and if μ is lower, the particle number is more likely to be larger.
- Let $z = q^{-\mu+1/2}$. Then the quantity for the grand canonical ensemble $F(z; q)$ is $p_0(z)f_0 + p_1(z)f_1 + p_2(z)f_2 + \cdots$, where p_k is the probability that the particle number is n. We furthermore have

$$p_k(z) = \frac{q^{(\binom{k}{2}}}{(q;q)_n} \frac{z^k}{(-z;q)_\infty}.$$

Then

$$f_n = q^{-\binom{k}{2}} (q;q)_n \frac{1}{2\pi i} \oint_0 \frac{(-z;q)_\infty}{z^n} F(z;q) \frac{dz}{z}.$$

Relation to nonintersecting Ornstein–Uhlenbeck processes

The Ornstein–Uhlenbeck (OU) process is a diffusion process defined by the stochastic differential equation

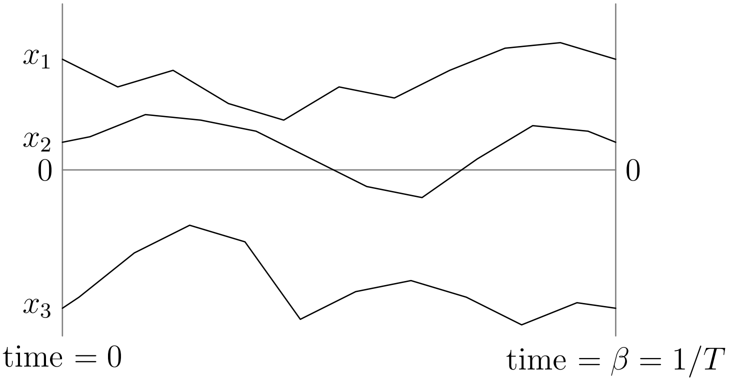
$$dx(t) = -x(t)dt + dW(t),$$

and can be understood as the Brownian motion in a quadratic potential well.

- Suppose n particles are in independent OU processes, starting from near 0 at time $-M$. If they are conditioned to be nonintersecting, and to end near 0 at time M, we have that at time 0, their distribution is the GUE (temperature $T = 0$ MNS matrix model) eigenvalue distribution.
- Suppose n particles are in independent OU processes, starting from random positions $x_1, \dots, x_n$ at time 0. If they are conditioned that after time $\beta = 1/T$, they return to positions $x_1, \dots, x_n$ again, and they are nonintersecting in time $[, 0, \beta]$, then in any time between $[0, \beta]$, we can compute the distribution of the particles.

Relation to nonintersecting OU processes (continued)

- If we require the time periodic nonintersecting OU processes to be **stationary**, that is, the distribution of particles do not depend on time in $[0, \beta]$, then the stationery distribution is the MNS matrix eigenvalue distribution at temperature T.
- What is the multi-time correlation?



Relation to nonintersecting OU processes (continued)

- The multi-time correlation function of the time periodic nonintersecting OU process is the same as the multi-time correlation function of free fermions in the canonical ensemble **along the imaginary time** (Le Doussal-Majumdar-Schehr).
- If we consider the grand canonical ensemble, the multi-time correlation function of the free fermions along the imaginary time is obtained by Le Doussal, Majumdar and Schehr, and it is a determinant. Since we do not go into the detail, we denote it simply as $F(z;q)$, where $z = q^{-\mu+1/2}$. Then the multi-time correlation function for the canonical ensemble is

$$q^{-\binom{n}{2}}(q;q)_n \frac{1}{2\pi i}\oint_0 \frac{(-z;q)_\infty}{z^n} F(z;q)\frac{dz}{z}.$$

An interesting identity

We also find an alternative proof for the gap probability formula. First, we have, by linear algebra, with the kernel

$$M(x, y; z) = \sum_{k=0}^{\infty} q^k \varphi_k(x)\varphi_k(y),$$

the identity

$$\det(I + zM(x, y; q)\chi_A) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{q^{n(n-1)/2}(q;q)_n} P(\text{all particles are in } A),$$

and then

$$P(\text{all particles are in } A) = q^{n(n-1)/2}(q;q)_n \frac{1}{2\pi i} \oint_0 \det(I + zM(x, y; q)\chi_A) \frac{dz}{z^{n+1}}.$$

An interesting identity

Then we have the following lemma for an integral operator C and its resolvent B ($I - B = (I + C)^{-1}$) on $L^2(\mathbb{R})$.

Lemma

As operators on $L^2(\mathbb{R})$,

$$I + C\chi_A = (I + C)(I - B\chi_{A^c}).$$

and then

$$\det(I + C\chi_A) = \det(I + C)\det(I - B\chi_{A^c}).$$

If $C = zM(q)$, its resolvent is $K(z; q)$. Using that $\det(I + zM(q)) = (-z; q)_\infty$, we prove the algebraic formula for gap probability.

Relation to KPZ universality class

- Kadar–Parisi–Zhang universality class contains many models in non-equilibrium statistical physics. A hallmark of models in this universality class is that the limiting distribution of a particle (at a fixed time) is the Tracy–Widom distribution. Moreover, as the transition between the KPZ universality class and the EW (Edwards-Wilkinson) universality class in which the limiting distribution of a particle is the Gaussian distirbution, we see the **crossover** distribution given by

$$\det(1 - K_{\text{crossover}}\chi_{(x,+\infty)}).$$

- Dean, Le Dousal, Majumdar, and Schehr observed that the free fermion model has asymptotics in the KPZ universality. In a different setting, Imamura and Sasamoto also noticed that another model in KPZ universality has features of the free fermion model. **Is it a coincidence?**

Analogy

Let $f(\eta)$ be a meromorphic function with poles $\{a_1, \dots, a_m\} = \mathbb{A}$ and $f(0) = 1$. Let $\Gamma_{\mathbb{A}}$ and $\Gamma_{0,\mathbb{A}}$ be contours below, such that $\Gamma_{\mathbb{A}}$ contains $\mathbb{A}$ but not 0, while $\Gamma_{0,\mathbb{A}}$ contains $\Gamma_{\mathbb{A}}$ as well as 0. Define

$$M(\xi, \eta; q) = \frac{f(\eta)}{\xi - q\eta},$$

$\mathbf{M}(q)$ be the integral operator defined on the contour $\Gamma = \Gamma_{0,\mathbb{A}} \cup (-\Gamma_{\mathbb{A}})$ with kernel $M(\xi, \eta; q)$, and $K(\xi, \eta; z; q)$ be the kernel of $\mathbf{K}(z; q)$ that is the resolvent of $z\mathbf{M}(q)$ on Γ, such that

$$I - \mathbf{K}(z; q) = (I + z\mathbf{M}(q))^{-1}.$$

Then we have

$$\det(I + zM(q)_{L^2(\Gamma_{0,\mathbb{A}})}) = (-z; q)_\infty \det(I + K(z; q)_{L^2(\Gamma_{\mathbb{A}})}),$$
$$\det(I + zM(q)_{L^2(\Gamma_{\mathbb{A}})}) = (-z; q)_\infty \det(I + K(z; q)_{L^2(\Gamma_{0,\mathbb{A}})}).$$

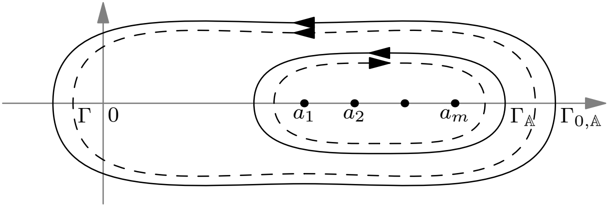
Formally,

$$M(\xi, \eta; q) = \sum_{k=0}^{\infty} q^k \varphi_k(\xi)\psi_k(\eta), \quad \varphi_k(\xi) = \xi^{-k-1}, \quad \psi_k(\eta) = f(\eta)\eta^k.$$

(Unfortunatedly, φ_k and ψ_k are not biorthogonal.)
Then formally where

$$K(\xi, \eta; z; q) = \sum_{k=1}^{\infty} (-1)^{k+1} z^k \frac{F(\eta; q; k)}{\xi - q^k \eta}, \quad F(\eta; q; k) := \prod_{j=0}^{k-1} f(q^j \eta).$$

Often $K(\xi, \eta; z; q)$ has a contour integral representation.



Case study: ASEP

The Asymetric Simple Exclusion Process is the most well-studied particle model in the KPZ universality class. Suppose each particle independently jumps to the right with rate $p \in (0, 1/2)$ and to the left with rate $q = 1 - p$, and initially infinitely many particles are randomly put on $\mathbb{Z}_+ = \{1, 2, \dots\}$, such that each site has probability $\rho \in (0, 1]$ to be occupied (step Bernoulli initial condition).

It is first obtained by Tracy and Widom by coordinate Bethe ansatz that the distribution of the m-th leftmost particle is

$$\mathbb{P}(x_m(t) \le x) = \frac{1}{2\pi i} \oint \frac{\det(I + z\mathbf{M}(\tau))_{L^2(\Gamma_{\mathbb{A}})}}{(-z;\tau)_m} \frac{dz}{z},$$

where $\mathbf{M}(q)$ is defined by

$$f(\eta) = \left(\frac{1+\eta}{1+\eta/\tau}\right)^x e^{-\frac{1-\tau}{1+\tau} t \left(\frac{1}{1+\eta/\tau} - \frac{1}{1+\eta}\right)} \frac{1}{1 - \frac{\eta}{\theta\tau}}, \quad \text{where} \quad \theta = \frac{\rho}{1-\rho}.$$

Then we have

$$\mathbb{P}(x_m(t) \le x) = \frac{1}{2\pi i} \oint (-zq^m; \tau)_\infty \det(I + \mathbf{K}(z;\tau))_{L^2(\Gamma_{0,\mathbb{A}})} \frac{dz}{z},$$

where

$$K(\xi, \eta; z; \tau) = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \frac{\pi z^s}{\sin(\pi s)} \frac{g(\eta)}{g(\tau^s \eta)} \frac{ds}{\xi - \tau^s \eta},$$

and

$$g(\eta) = (1 + \eta/\tau)^{-x} e^{\frac{1-\tau}{1+\tau} \frac{t}{1+\eta/\tau}} \frac{1}{(\eta/(\theta\tau), \tau)_\infty}.$$

This formula is equivalent to a formula derived by Borodin-Corwin-Sasamoto by the q-moment method.

Case study: q-TAZRP

The q-deformed Totally Asymmetric Zero Range Process (q-TAZRP) is another well-studied particle model in the KPZ universality class. Suppose at each site, the top particle jumps to the right at rate $(1-q^h)/(1-q)$ where h is the height of the stack of particles, and suppose initially N particles are at site 0. Then by coordinate Bethe ansatz, the distribution of the leftmost particle is

$$\mathbb{P}_{0^N}(x_N(t) \geq M) = [N]_q! \frac{(q-1)^N}{q^{N(N-1)/2}} \frac{1}{2\pi i} \oint_0 \det(1 + z\mathbf{M}(q)_{L^2(\Gamma_{0,\mathbb{A}})}) \frac{dz}{z^{N+1}},$$

where $\mathbf{M}(q)$ is defined by

$$f(\eta) = \left(\frac{1}{1-\eta}\right)^M e^{-\eta t}.$$

Then we have

$$\mathbb{P}_{0^N}(x_N(t) \geq M) =$$
$$[N]_q! \frac{(q-1)^N}{q^{N(N-1)/2}} \frac{1}{2\pi i} \oint_0 \frac{(-z;q)_\infty}{z^{N+1}} \det(1 + \mathbf{K}(z;q))_{L^2(\Gamma_{\mathbb{A}})} \, dz,$$

where

$$K(\xi, \eta; z; q) = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \frac{\pi z^s}{\sin(\pi s)} \frac{g(\eta)}{g(q^s \eta)} \frac{ds}{\xi - q^s \eta},$$

and

$$g(\eta) = e^{-\frac{t}{1-q}\eta} \frac{1}{(\eta; q)_\infty^M}.$$

The formal similarity with the gap probability formula for free fermions at finite temperature may suggest that the particle model has also a fermionic intepretation. But so far we have not identified it.