Blocks in the Asymmetric Simple Exclusion Process

Painlevé Equations and Applications:
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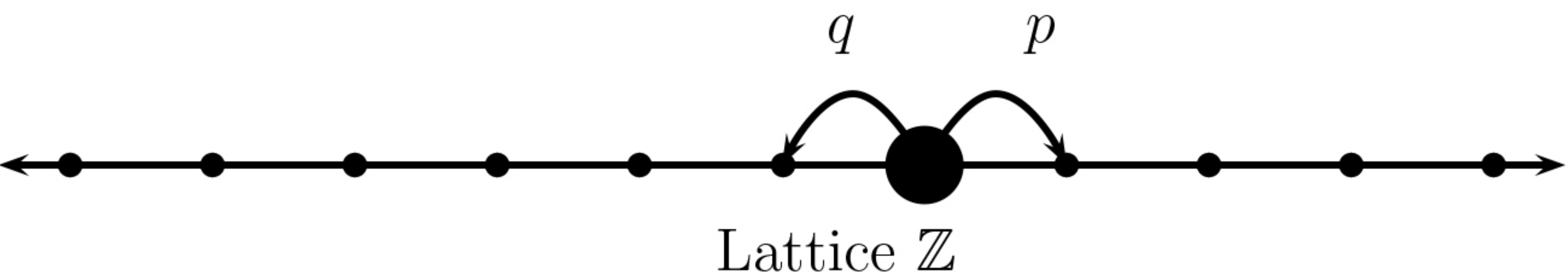
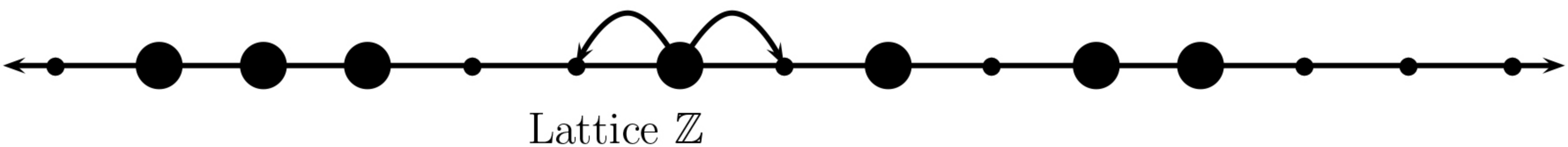
Simple Random Walk



- Can make **time continuous** by giving particle a “random alarm clock”, i.e. exponential distr. with mean 1.
- This is arguably one of the most important, if elementary, *stochastic processes*.
- Want **many particles**—to be interesting these particles must interact.

Asymmetric Simple Exclusion Process (ASEP)

A continuous time Markov process



- Particles move on $\mathbb{Z}$ according to two rules:
  1. A particle waits at *x* an exponential time with parameter one, and then chooses *y* with probability *p(x,y).*
  2. If *y* is vacant at that time it moves to *y,* while if *y* is occupied it remains at *x.*
- *"Simple"* refers to the fact that jumps are allowed only one step to either the right or left
- *"Asymmetric"* refers to the case $p \neq q$.

Transition Probability: $P_Y(x;t)$

For **one particle** the probability that the particle is initially at *y* is at *x* at time *t* is

$$P_y(x;t) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}_r} \xi^{x-y-1} \mathrm{e}^{t\varepsilon(\xi)} \, d\xi$$

where

$$\varepsilon(\xi) = \frac{p}{\xi} + q\xi - 1$$

and $\mathcal{C}_r$ is a circle of radius r centered at the origin.

This result is elementary but the generalization to more than one particle is rather subtle

N-particle ASEP

Initial configuration: $Y := \{y_1, y_2, \dots, y_N\}$ with $y_1 < y_2 < \cdots < y_N$.

Final configuration: $X := \{x_1, x_2, \dots, x_N\}$ with $x_1 < x_2 < \cdots < x_N$.

Let $\mathfrak{S}_N$ denote the permutation group and set

$$\begin{aligned} U(\xi, \xi') &= \frac{p + q\,\xi\,\xi' - \xi}{\xi' - \xi} \\ A_\sigma(\xi) &= \prod_{1 \le i < j \le N} \frac{U\left(\xi_{\sigma(i)}, \xi_{\sigma(j)}\right)}{U\left(\xi_i, \xi_j\right)}, \quad \sigma \in \mathfrak{S}_N. \end{aligned}$$

Theorem (TW, 2008).

$$P_Y(X;t) = \sum_{\sigma \in \mathfrak{S}_N} \int_{\mathcal{C}_r} \cdots \int_{\mathcal{C}_r} A_\sigma(\xi) \prod_{j=1}^{N} \xi_{\sigma(j)}^{x_j - y_{\sigma(j)} - 1} \mathrm{e}^{t\varepsilon(\xi_j)}\, d\xi_1 \cdots d\xi_N$$

where $\mathcal{C}_r$ has radius so small that all the poles of A_σ lie outside of $\mathcal{C}_r$.

Remarks:

- $P_Y(X;t)$ satisfies $P_Y(x;0) = \delta_{X,Y}$.
- This is a sum of $N!$ terms with each term an N-dimensional contour integral.
- We are ultimately interested in $N \to \infty$. Not at all clear how to proceed!

- To extract information from $P_Y(x;t)$, we start by looking at *marginal distributions;* the simplest are one-point functions:

$$\mathbb{P}_Y\left(x_m(t) = x\right)$$

Must sum $P_Y(X;t)$ over all configurations satisfying $x_m(t) = x$.

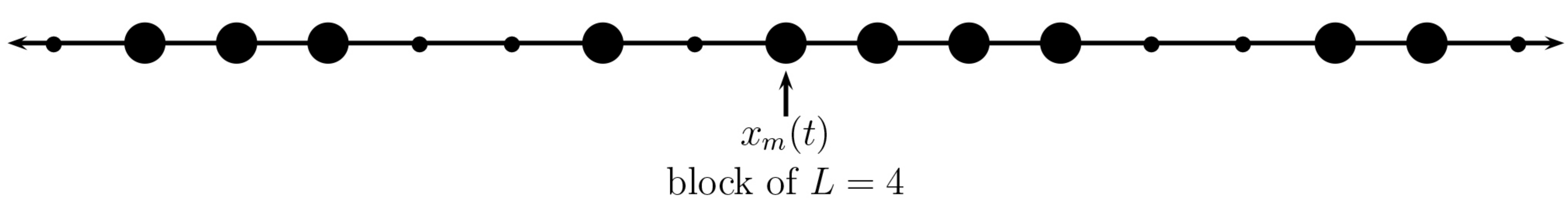
For example, for $m = 2$ we must sum over configurations X

$$X = \{x - v_1, x, x + v_2, x + v_2 + v_3, \ldots, x + v_2 + v_3 + \cdots + v_N\}$$

where $v_i = 1, 2, 3, \ldots$

Second Example: **ASEP Blocks**

*m*th particle is the left-most one in a contiguous block of *L* particles



Case *m=1, left-most particle*
Identity One

Identity One. For $N \geq L$,

$$\sum_{\sigma \in \mathfrak{S}_N} \prod_{1 \leq i,j \leq N} U\left(\xi_{\sigma(i)}, \xi_{\sigma(j)}\right) \frac{\xi_{\sigma(2)} \xi_{\sigma(3)}^2 \cdots \xi_{\sigma(N)}^{N-1}}{(1 - \xi_{\sigma(L+1)} \cdots \xi_{\sigma(N)}) \cdots (1 - \xi_{\sigma(N-1)} \xi_{\sigma(N)})(1 - \xi_{\sigma(N)})}$$

$$= p^{N(N-1)/2} \frac{\mathfrak{f}_L(\xi)}{\prod_i (1 - \xi_i)}$$

where $\mathfrak{f}_L(\xi)$ are *symmetric polynomials* in the variables $\xi = (\xi_1, \ldots, \xi_N)$.

For the definition of $\mathfrak{f}_L(\xi)$ we first define

$$\varphi_L(z_1, \ldots, z_L; \xi) = \frac{\prod_{1 \leq j \leq N} U(z_1, \xi_j)\, U(z_2, \xi_j) \cdots U(z_L, \xi_j)}{z_1^L\, (q z_1 - p)\, z_2^{L-1}\, (q z_2 - p) \cdots z_L\, (q z_L - p)} \prod_{1 \leq i < j \leq L} \frac{1}{U(z_j, z_i)}$$

$$\text{then} \quad \mathfrak{f}_L(\xi) = p^{L(L+1)/2 - LN} \prod_i \xi_i^L \int_{\Gamma_\xi} \cdots \int_{\Gamma_\xi} \varphi_L(z_1, \ldots, z_L; \xi)\, dz_1 \cdots dz_L,$$

Γ_ξ consists of simple closed curves enclosing the points ξ_j but no other singularities of the integrand.

For $L = 1$,

$$\mathfrak{f}_1(\xi) = 1 - \prod_i \xi_i.$$

but the complexity of $\mathfrak{f}_L$ increases with L.

General *m*
Identity Two

Notation:

- S is a subset of $\{1, 2, \ldots, N\}$.
- $\widehat{\xi_S}$ denotes the variables ξ_k with $k \notin S$.
- Set $\tau := p/q < 1$ and recall the τ-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_\tau = \frac{(1-\tau^n)\cdots(1-\tau^{n-k+1})}{(1-\tau)\cdots(1-\tau^k)}$$

Identity Two: For $0 \leq m \leq N - L$,

$$\sum_{|S|=m} \prod_{\substack{i \in S \\ j \notin S}} U(\xi_i, \xi_j) \cdot \mathfrak{f}_L(\widehat{\xi_S}) = q^{m(N-m)} \begin{bmatrix} N-L \\ m \end{bmatrix}_\tau \mathfrak{f}_L(\xi)$$

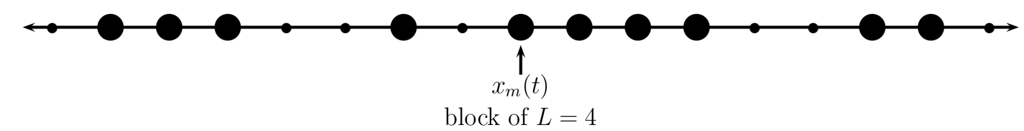
where $\mathfrak{f}_L(\xi)$ are the symmetric polynomials from *Identity One* and

$$U(\xi, \xi') = \frac{p + q\xi\xi' - \xi}{\xi' - \xi}.$$

What do the Identities buy for you?

Notation:

- $\mathcal{P}_{L,Y}(x,m,t)$: probability that at time t the mth particle from the left is the beginning of a block of length L starting at x.



- $$I_L(x,Y,\xi) := \prod_{1\le i<j\le N} \frac{1}{U(\xi_i,\xi_j)} \prod_i \frac{1}{1-\xi_i}\, \mathfrak{f}_L(\xi) \prod_i \left(\xi_i^{x-y_i-1} \mathrm{e}^{\varepsilon(\xi_i)t}\right)$$

- S a subset of $\{1,\dots,N\}$, S^c complement of S.

- $I_L(x,Y_S,\xi_S)$ indices lie in S.

- $\sigma(S^c)$ is the sum of the elements in S^c.

Theorem (TW, $L=1$, 2008; general L, 2017): For $q>0$

$$\mathcal{P}_{L,Y}(x,m,t) = p^{(N-m+1)(N-m)/2} q^{(m-1)(N-m/2)} \sum_{|S^c|<m} (-1)^{m-1-|S^c|} \begin{bmatrix} |S|-L \\ m-1-|S^c| \end{bmatrix}_\tau$$
$$\times \frac{q^{\sigma(S^c)-N|S^c|}}{p^{\sigma(S^c)-|S^c|(|S^c|+1)/2}} \int_{\mathcal{C}_r} \cdots \int_{\mathcal{C}_r} I_L(x,Y_S,\xi_S)\, d^{|S|}\xi$$

Remarks:

- The proof for general L proceeds exactly the same as for $L=1$ given the general L identities and the fact that $\mathfrak{f}_L(\xi)$ are polynomials—no new poles introduced in the argument.

- As was the case for $L=1$, there is a formula for $\mathcal{P}_{L,Y}(x,m,t)$ but with integrations over large contours. In this expression one can let $N\to\infty$.

Large contour representation

Notation:

- $\mathcal{P}_{L,Y}(x,m,t)$: probability that at time t the mth particle from the left is the beginning of a block of length L starting at x.

- $$I_L(x,Y,\xi) := \prod_{1\le i<j\le N} \frac{1}{U(\xi_i,\xi_j)} \prod_i \frac{1}{1-\xi_i}\, \mathfrak{f}_L(\xi) \prod_i \left(\xi_i^{x-y_i-1} \mathrm{e}^{\varepsilon(\xi_i)t}\right)$$

- S a subset of $\{1,\ldots,N\}$.

- $I_L(x,Y_S,\xi_S)$ indices lie in S.

- $\sigma(S)$ is the sum of the elements in S.

Theorem (TW, $L=1$, 2008; general L, 2017): For $q>0$

$$\begin{aligned}\mathcal{P}_{L,Y}(x,m,t) &= (-1)^{m+1} p^{m(m-1)/2} \sum_{|S|\ge m+L-1} q^{(m-1)(|S|-m/2)} \begin{bmatrix} |S|-L \\ m-1 \end{bmatrix}_\tau \\ &\quad \times \frac{p^{\sigma(S)-m|S|}}{q^{\sigma(S)-|S|(|S|+1)/2}} \int_{\mathcal{C}_R} \cdots \int_{\mathcal{C}_R} I_L(x,Y_S,\xi_S)\, d^{|S|}\xi\end{aligned}$$

where R is so large that the poles of the integrand lie inside $\mathcal{C}_R$.

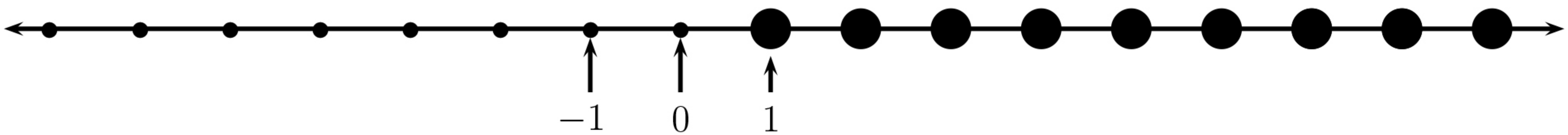
Remarks:

- This theorem extends to **infinite systems unbounded on the right**. The sum is then taken over finite subsets of $\mathbb{Z}^+$.

- Up to this point the initial configuration $Y=\{y_1,y_2,\ldots\}$, $y_1<y_2<\cdots$, is completely general (though bounded below). We now turn to the special case of *step initial condition.*

Step Initial Condition

Drift to the left, $p < q$
Particles initially occupy $\mathbb{Z}^+$



Remarks:

- In the stochastic growth interpretation of ASEP, the step initial condition corresponds to the *droplet initial condition*.
- We are interested in $\mathcal{P}_{L,\mathbb{Z}^+}(x,m,t)$.
- One starts with the *large contour representation* of $\mathcal{P}_{L,\mathbb{Z}^+}(x,m,t)$, and then first sums over all S with $|S|$ equal to a fixed k.

Fredholm Determinant Representation

Notation:

- Denote by $K_{L,x}(z)$ the integral operator acting on functions on $\mathcal{C}_R$ with kernel

$$
\begin{aligned}
K_{L,x}(\xi,\xi';z) &= K_x(\xi,\xi')\prod_{j=1}^{L} U(z_j,\xi), \quad \text{where} \\
K_x(\xi,\xi') &= \frac{\xi^x \, \mathrm{e}^{\varepsilon(\xi)t}}{p+q\,\xi\,\xi'-\xi}
\end{aligned}
$$

- τ-Pochhammer symbol, $(\lambda;\tau)_m := \prod_{j=0}^{m-1}(1-\lambda\tau^j)$.

Theorem (TW, $L=1$, 2008; general L, 2017). For $p,q>0$,

$$
\mathcal{P}_{L,\mathbb{Z}^+}(x,m,t) = (-1)^{L-1}\,p^{L(L+1)/2}\,\tau^{-(m-1)(L-1)}
$$

$$
\times \int_{\Gamma_{0,\tau}} \cdots \int_{\Gamma_{0,\tau}} \frac{1}{z_1^L\,(qz_1-p)\,z_2^{L-1}\,(qz_2-p)\cdots z_L\,(qz_L-p)} \prod_{i<j}\frac{1}{U(z_j,z_i)}
$$

$$
\times \left[\int \frac{\det(I-p^{-L}q\,\lambda\,K_{L,\,x+L-1}(z))}{(\lambda;\tau)_m}\,\frac{d\lambda}{\lambda^L}\right] dz_L\cdots dz_1\,.
$$

Remarks:

- The z-iterated integral is interpreted as follows: First take the sum of the residues at $z_L=0$ and $z_L=\tau$. In the resulting integrand take the sum of the residues at $z_{L-1}=0$ and $z_{L-1}=\tau$; and so on.

- The λ-integration is over a contour enclosing the singularities of the integrand at τ^{-j} for $j=0,\ldots,m-1$.

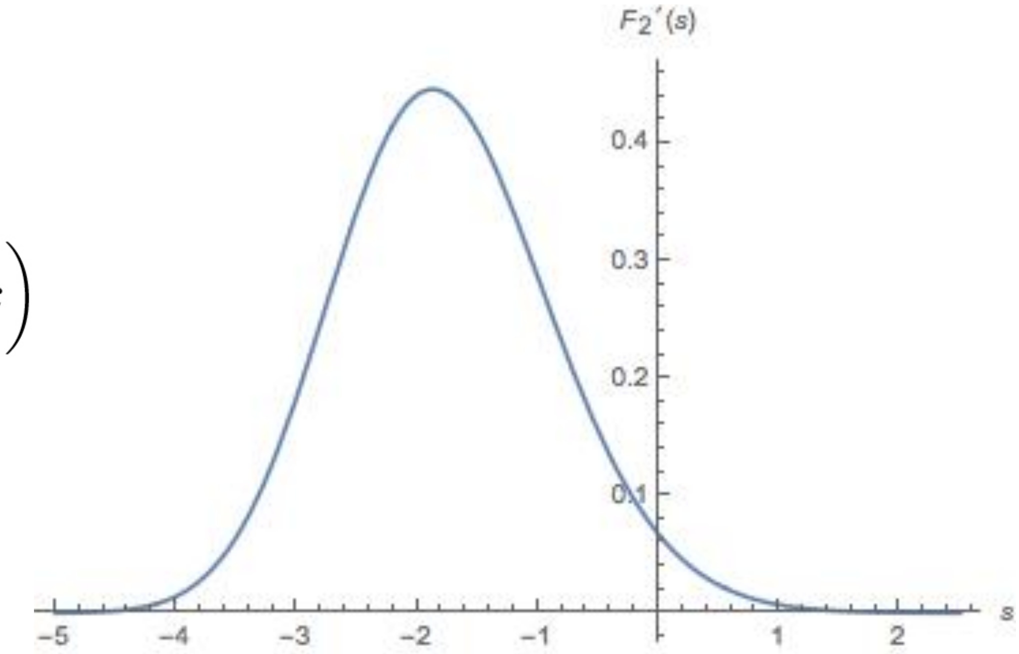
- For $L=1$, evaluating the z_1-integral leads to the result

$$
\mathbb{P}_{\mathbb{Z}^+}(x_m(t)\le x) = \int \frac{\det(I-q\lambda K_x)}{(\lambda;\tau)_m}\,\frac{d\lambda}{\lambda}
$$

  which is the 2008 result.

But what does all this ASEP stuff have to do with Painlevé?

A Limit Theorem



Notation:

- $$F_2(s) := \exp\left(-\int_s^\infty (x-s)q^2(x)\,dx\right)$$

  where q is the Hastings-McLeod solution of P_{II}

  $$\frac{d^2q}{dx^2} = xq + 2q^3.$$

- $$\sigma = \frac{m}{t}, \quad , c_1 = -1 + 2\sqrt{\sigma}, \;\; c_2 = \sigma^{-1/6}(1-\sqrt{\sigma})^{2/3}, \;\; \gamma = q - p,$$

Theorem (TW, $L=1$, 2009). When $0 \leq p < q$ we have

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{x_m(t/\gamma) - c_1 t}{c_2 t^{1/3}} \leq s\right) = F_2(s)$$

uniformly for σ in a compact subset of $(0,1)$.

Remarks:

- For $p = 0$ (TASEP), this result was first proved by Johansson (2000).
- The above Theorem was the first "KPZ Universality Theorem" outside the class of *determinantal processes*. (TASEP is a determinantal process.)
- The proof required a deformation of the K_x-operator to a new J-operator that (essentially) left the Fredholm determinant of K_x unchanged. The Fredholm determinant of the J-operator is amendable to asymptotic analysis.
- The J-operator representation was the starting point for the Amir-Corwin-Quastel and (independently) the Sasamoto-Spohn analysis of the KPZ crossover scaling function.

Future Work

- Extend the *J-kernel* analysis to general *L.* (Essentially done.)
- Carry out an asymptotic analysis in the KPZ limit. (Work in progress.)
- Develop a better understanding of the identities. For *L=1* a deeper interpretation was found by Borodin, Corwin, Petrov and Sasamoto (Commun. Math. Phys. 329 (2015), 1167—1245.
- "ASEP Gaps" (work in progress)

Thank you for your attention