Nonintersecting Brownian motions on the unit circle

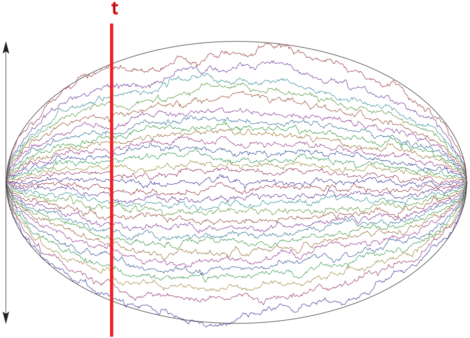
Karl Liechty
joint work with Dong Wang and Robert Buckingham

DePaul University

Painlevé Equations and Applications: A Workshop in Memory of A. A. Kapaev
August 27, 2017

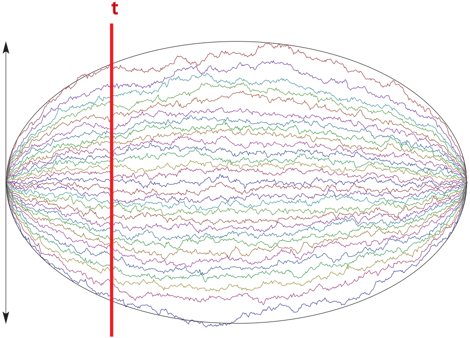
Nonintersecting paths and determinantal processes

Consider an ensemble of n Brownian bridges conditioned not to intersect.



Nonintersecting paths and determinantal processes

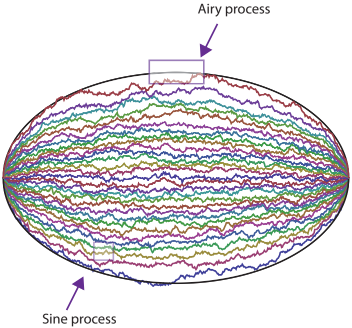
Consider an ensemble of n Brownian bridges conditioned not to intersect.



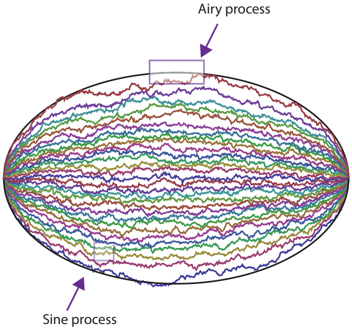
For each fixed time t, the locations of particles is a *determinantal point process* (DPP), i.e., there is some *kernel function* $K(x, y)$ such that

$$\lim_{\Delta x \to 0} \frac{1}{(\Delta x)^m} \mathbb{P}(\text{there are particles in } [x_1 + \Delta x], \dots, [x_m + \Delta x]) = \det \left(K(x_i, x_j) \right)_{i,j=1}^{m}.$$

Nonintersecting Brownian bridges



In the simplest model in which all particles begin at 0 at time $t = 0$ and end at 0 at time $t = 1$, The correlation kernel is then given in terms of Hermite polynomials. In fact, the distribution of particles for a fixed time is exactly the same as the Gaussian Unitary Ensemble of random matrices



In the simplest model in which all particles begin at 0 at time $t = 0$ and end at 0 at time $t = 1$, The correlation kernel is then given in terms of Hermite polynomials. In fact, the distribution of particles for a fixed time is exactly the same as the Gaussian Unitary Ensemble of random matrices

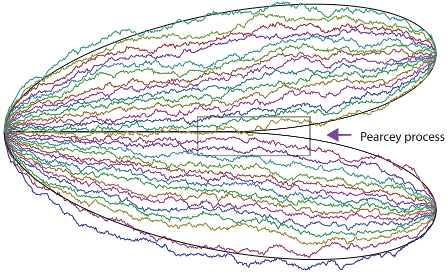
As $n \to \infty$, the paths collect inside an ellipse. The local scaling limit in the bulk is the *sine process*, and at the edge is the *Airy process*.

Other limiting processes

Other universal processes can be observed if we change the model a bit.

Other limiting processes

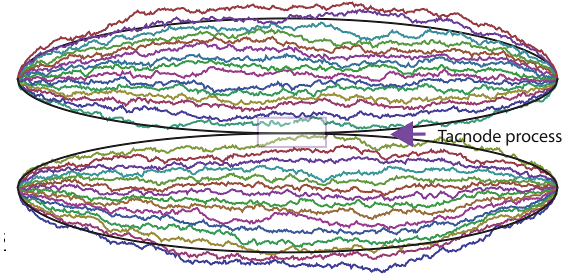
Other universal processes can be observed if we change the model a bit. Two distinct ending points:



The kernel is now described in terms of *multiple orthogonal polynomials*. The local scaling limit near the cusp is the *Pearcey process*, so called because the extended kernel is in terms of Pearcey integrals (Brézin–Hikami 1998, Bleher–Kuijlaars 2005, Tracy–Widom 2005)

Other limiting processes

Two starting and two ending points:



The kernel is described in terms of *multiple (Hermite) orthogonal polynomials*. The local scaling limit near the tacnode is the *tacnode process*, which in some sense describes two Airy processes which come together but do not intersect. Delvaux–Kuijlaars–Zhang, Adler–Ferrari–Van Moerbeke, 2010; Johansson 2011.

Deformations of these processes

- k-Airy process (Adler–van Moerbeke–Ferrari 2010) :

$$K_{k\mathrm{Airy}}(x,y;t) = \left(\frac{1}{2\pi i}\right)^2 \int_{\tilde{\mathcal{C}}} du \int_{\mathcal{C}} dv \left(\frac{v+t}{u+t}\right)^k \frac{e^{\frac{1}{3}(u^3-v^3)+xu-yv}}{v-u}$$

Deformations of these processes

- k-Airy process (Adler–van Moerbeke–Ferrari 2010) :

$$K_{k\mathrm{Airy}}(x,y;t) = \left(\frac{1}{2\pi i}\right)^2 \int_{\tilde{\mathcal{C}}} du \int_{\mathcal{C}} dv \left(\frac{v+t}{u+t}\right)^k \frac{e^{\frac{1}{3}(u^3-v^3)+xu-yv}}{v-u}$$

- Pearcey process with k inliers Adler–Delephine–van Moerbeke 2011):

$$K_{k\mathrm{Pearcey}}(x,y;t) = \left(\frac{1}{2\pi i}\right)^2 \int_X du \int_{-i\cdot\infty}^{i\cdot\infty} dv \frac{e^{\frac{u^4}{4}-\frac{tu^2}{2}+xu}}{e^{\frac{v^4}{4}-\frac{tv^2}{2}+yv}} \frac{1}{v-u} \left(\frac{v}{u}\right)^k$$

Deformations of these processes

- k-Airy process (Adler–van Moerbeke–Ferrari 2010) :

$$K_{k\mathrm{Airy}}(x,y;t) = \left(\frac{1}{2\pi i}\right)^2 \int_{\tilde{\mathcal{C}}} du \int_{\mathcal{C}} dv \left(\frac{v+t}{u+t}\right)^k \frac{e^{\frac{1}{3}(u^3-v^3)+xu-yv}}{v-u}$$

- Pearcey process with k inliers Adler–Delephine–van Moerbeke 2011):

$$K_{k\mathrm{Pearcey}}(x,y;t) = \left(\frac{1}{2\pi i}\right)^2 \int_X du \int_{-i\cdot\infty}^{i\cdot\infty} dv \frac{e^{\frac{u^4}{4}-\frac{tu^2}{2}+xu}}{e^{\frac{v^4}{4}-\frac{tv^2}{2}+yv}} \frac{1}{v-u} \left(\frac{v}{u}\right)^k$$

- Tacnode process: there is a double contour integral formula, but integrand is not made up of elementary functions

Deformations of these processes

- k-Airy process (Adler–van Moerbeke–Ferrari 2010) :

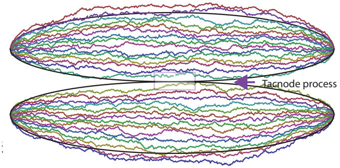
$$K_{k\mathrm{Airy}}(x,y;t) = \left(\frac{1}{2\pi i}\right)^2 \int_{\tilde{\mathcal{C}}} du \int_{\mathcal{C}} dv \left(\frac{v+t}{u+t}\right)^k \frac{e^{\frac{1}{3}(u^3-v^3)+xu-yv}}{v-u}$$

- Pearcey process with k inliers Adler–Delephine–van Moerbeke 2011):

$$K_{k\mathrm{Pearcey}}(x,y;t) = \left(\frac{1}{2\pi i}\right)^2 \int_X du \int_{-i\cdot\infty}^{i\cdot\infty} dv \frac{e^{\frac{u^4}{4}-\frac{tu^2}{2}+xu}}{e^{\frac{v^4}{4}-\frac{tv^2}{2}+yv}} \frac{1}{v-u} \left(\frac{v}{u}\right)^k$$

- Tacnode process: there is a double contour integral formula, but integrand is not made up of elementary functions
- In each case, "shape" of kernel is amenable to Bertola–Caffaso-type analysis (Girotti's talk yesterday)

The tacnode process



- Adler–Ferrari–Van Moerbeke (2010): Double contour integral of Airy functions and related operators from discrete model
- Delvaux–Kuijlaars–Zhang (2010): A solution to a certain 4×4 Riemann–Hilbert problem, or equivalently, the solution to a 4×4 Lax system for the Painlevé II equation.
- Johansson (2011): Double contour integral of Airy functions and related operators from NIBM
- Ferrari–Vető (2012): Introduced asymmetric tacnode process, formula similar to Johansson
- L–Wang (2016): double contour integral formula for symmetric tacnode kernel with integrand made from solutions to 2×2 Lax pair for PII
- L–Wang (2016): 2×2 Lax pair extended to asymmetric kernel

A double contour integral formula for tacnode kernel

The Hastings–McLeod solution to the homogeneous Painlevé II equation (PII), $u_{\mathrm{HM}}(s)$, is the solution to the differential equation

$$u''(s) = su(s) + 2u(s)^3 \,,$$

which satisfies

$$u_{\mathrm{HM}}(s) = \mathrm{Ai}(s)(1 + o(1)) \,, \qquad \text{as } s \to +\infty \,,$$

where $\mathrm{Ai}(s)$ is the Airy function.

A double contour integral formula for tacnode kernel

The Hastings–McLeod solution to the homogeneous Painlevé II equation (PII), $u_{HM}(s)$, is the solution to the differential equation

$$u''(s) = su(s) + 2u(s)^3 \,,$$

which satisfies

$$u_{HM}(s) = \mathrm{Ai}(s)(1 + o(1)) \,, \qquad \text{as } s \to +\infty \,,$$

where $\mathrm{Ai}(s)$ is the Airy function.
Now consider the 2×2 matrix-valued solutions to the differential equation

$$\frac{d}{d\zeta}\mathbf{\Psi}(\zeta; s) = \begin{pmatrix} -4i\zeta^2 - i(s + 2u_{HM}(s)^2) & 4\zeta u_{HM}(s) + 2iu'_{HM}(s) \\ 4\zeta u_{HM}(s) - 2iu'_{HM}(s) & 4i\zeta^2 + i(s + 2u_{HM}(s)^2) \end{pmatrix} \mathbf{\Psi}(\zeta; s) \,. \qquad (1)$$

This differential equation, together with another one with respect to s, form a Lax pair for the PII equation, i.e., the compatibility of the two differential equations implies that $u_{HM}(s)$ solves PII.

A double contour integral formula for tacnode kernel

The Hastings–McLeod solution to the homogeneous Painlevé II equation (PII), $u_{\mathrm{HM}}(s)$, is the solution to the differential equation

$$u''(s) = su(s) + 2u(s)^3\,,$$

which satisfies

$$u_{\mathrm{HM}}(s) = \mathrm{Ai}(s)(1 + o(1))\,, \qquad \text{as } s \to +\infty\,,$$

where $\mathrm{Ai}(s)$ is the Airy function.
Now consider the 2×2 matrix-valued solutions to the differential equation

$$\frac{d}{d\zeta}\mathbf{\Psi}(\zeta; s) = \begin{pmatrix} -4i\zeta^2 - i(s + 2u_{\mathrm{HM}}(s)^2) & 4\zeta u_{\mathrm{HM}}(s) + 2iu'_{\mathrm{HM}}(s) \\ 4\zeta u_{\mathrm{HM}}(s) - 2iu'_{\mathrm{HM}}(s) & 4i\zeta^2 + i(s + 2u_{\mathrm{HM}}(s)^2) \end{pmatrix} \mathbf{\Psi}(\zeta; s)\,. \tag{1}$$

This differential equation, together with another one with respect to s, form a Lax pair for the PII equation, i.e., the compatibility of the two differential equations implies that $u_{\mathrm{HM}}(s)$ solves PII.
We consider the particular solution to (1) which satisfies

$$\mathbf{\Psi}(\zeta; s)e^{i(\frac{4}{3}\zeta^3 + s\zeta)\sigma_3} = I + O(\zeta^{-1})\,, \quad \zeta \to \pm\infty\,.$$

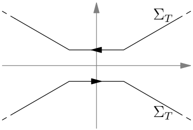
A double contour integral formula for tacnode kernel

Define the functions f and g in terms of the entries of the matrix $\Psi(u;s)$:

$$f(u;s) := \begin{cases} -\Psi_{12}(u;s) & \text{if } \operatorname{Im} u > 0, \\ \Psi_{11}(u;s) & \text{if } \operatorname{Im} u < 0, \end{cases} \qquad g(u,s) := \begin{cases} -\Psi_{22}(u;s) & \text{if } \operatorname{Im} u > 0, \\ \Psi_{21}(u;s) & \text{if } \operatorname{Im} u < 0. \end{cases}$$

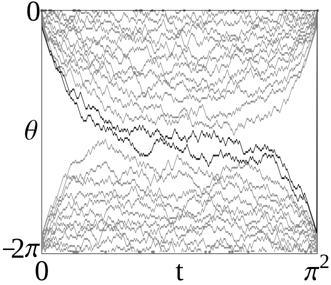
The tacnode kernel is then given as (L–Wang '16)

$$K_{\mathrm{tac}}(\xi,\eta;t,\sigma) := \frac{1}{2\pi}\int_{\Sigma_T} du \int_{\Sigma_T} dv\, e^{\frac{t}{2}(u^2-v^2)} e^{-i(u\xi - v\eta)} \frac{f(u;\sigma)g(v;\sigma) - g(u;\sigma)f(v;\sigma)}{2\pi i(u-v)}$$



A deformation of the tacnode process

We could also consider a version of the tacnode process in which a finite number of walkers "switch sides".



We call this process the *k*-**tacnode process**.

The generalized Hastings–McLeod solution to PII

Consider the solution, $u_{\text{HM}}^{(\alpha)}$, to the inhomogeneous Painlevé-II equation

$$u''(s) = 2u(s)^3 + su(s) - \alpha$$

satisfying both

$$u_{\text{HM}}^{(\alpha)}(s) = \sqrt{\frac{-s}{2}}\left(1 + \mathcal{O}\left(\frac{1}{(-s)^{3/2}}\right)\right) \text{ as } s \to -\infty$$

and

$$u_{\text{HM}}^{(\alpha)}(s) = \frac{\alpha}{s}\left(1 + \mathcal{O}\left(\frac{1}{s^3}\right)\right) \text{ as } s \to +\infty \quad (\alpha \neq 0).$$

This solution is called the (generalized) Hastings–McLeod solution (Its/Kapaev (2003), Claeys/Kuijlaars/Vanlessen (2008)).

The τ-functions for HM solution to PII

Now let $\mathcal{U}_k$ and $\mathcal{V}_k$ be the solutions of the coupled Painlevé-II system

$$\mathcal{U}_k''(s) = 2\mathcal{U}_k(s)^2\mathcal{V}_k(s) + s\mathcal{U}_k(s),$$
$$\mathcal{V}_k''(s) = 2\mathcal{U}_k(s)\mathcal{V}_k(s)^2 + s\mathcal{V}_k(s),$$

with asymptotic behavior

$$\mathcal{U}_k(s) = \begin{cases} \dfrac{-ik!}{2\cdot 8^k\sqrt{\pi}s^{(2k+1)/4}}e^{-\frac{2}{3}s^{3/2}}\left(1+\mathcal{O}\left(\dfrac{1}{s^{3/4}}\right)\right), & s\to+\infty, \\ -\dfrac{i(-s)^{(2k+1)/2}}{8^k\sqrt{2}}\left(1+\mathcal{O}\left(\dfrac{1}{(-s)^{3/2}}\right)\right), & s\to-\infty, \end{cases}$$

$$\mathcal{V}_k(s) = \begin{cases} \dfrac{8^k\sqrt{\pi}is^{(2k-1)/4}}{(k-1)!}e^{\frac{2}{3}s^{3/2}}\left(1+\mathcal{O}\left(\dfrac{1}{s^{3/4}}\right)\right), & s\to+\infty, \\ \dfrac{2^{3k-1}\sqrt{2}i}{(-s)^{(2k-1)/2}}\left(1+\mathcal{O}\left(\dfrac{1}{(-s)^{3/2}}\right)\right), & s\to-\infty. \end{cases}$$

The τ-functions for HM solution to PII

When $k = 0$, these functions are simply multiples of the Hastings–McLeod function for $\alpha = 0$:

$$\mathcal{U}_0(s) = -iu_{\text{HM}}(s), \quad \mathcal{V}_0(s) = iu_{\text{HM}}(s).$$

For $k > 0$, if we define

$$p_k(s) := \frac{\mathcal{U}_k'(s)}{\mathcal{U}_k(s)}, \quad q_k(s) := \frac{\mathcal{V}_k'(s)}{\mathcal{V}_k(s)}$$

and

$$P_k(s) := 2^{-1/3}p_k(-2^{-1/3}s), \quad Q_k(x) := 2^{-1/3}q_k(-2^{-1/3}s),$$

Then

$$P_k(x) \equiv -u_{\text{HM}}^{(k+\frac{1}{2})}(x), \quad Q_k(x) \equiv u_{\text{HM}}^{(k-\frac{1}{2})}(x).$$

A formula for the k-tacnode kernel

Let $\widetilde{\mathbf{L}}_k(\zeta; s)$ be the 2×2 matrix-valued function satisfying the differential equation (part of Jimbo–Miwa–Garnier pair)

$$\frac{\partial \widetilde{\mathbf{L}}_k}{\partial \zeta} = \left(-4i\sigma_3\zeta^2 + 4i \begin{bmatrix} 0 & \mathcal{U}_k(s) \\ -\mathcal{V}_k(s) & 0 \end{bmatrix} \zeta + \begin{bmatrix} -2i\mathcal{U}_k(s)\mathcal{V}_k(s) - is & -2\mathcal{U}_k'(s) \\ -2\mathcal{V}_k'(s) & 2i\mathcal{U}_k(s)\mathcal{V}_k(s) + is \end{bmatrix} \right) \widetilde{\mathbf{L}}_k,$$

with the boundary condition

$$\widetilde{\mathbf{L}}_k(\zeta; s)\zeta^{-k\sigma_3} e^{i(\frac{4}{3}\zeta^3 + s\zeta)\sigma_3} = \mathbb{I} + \mathcal{O}\left(\frac{1}{\zeta}\right), \qquad \text{as } \zeta \to \pm\infty.$$

A formula for the k-tacnode kernel

Let $\widetilde{\mathbf{L}}_k(\zeta;s)$ be the 2×2 matrix-valued function satisfying the differential equation (part of Jimbo–Miwa–Garnier pair)

$$\frac{\partial \widetilde{\mathbf{L}}_k}{\partial \zeta} = \left(-4i\sigma_3\zeta^2 + 4i\begin{bmatrix} 0 & \mathcal{U}_k(s) \\ -\mathcal{V}_k(s) & 0 \end{bmatrix}\zeta + \begin{bmatrix} -2i\mathcal{U}_k(s)\mathcal{V}_k(s) - is & -2\mathcal{U}_k'(s) \\ -2\mathcal{V}_k'(s) & 2i\mathcal{U}_k(s)\mathcal{V}_k(s) + is \end{bmatrix}\right)\widetilde{\mathbf{L}}_k,$$

with the boundary condition

$$\widetilde{\mathbf{L}}_k(\zeta;s)\zeta^{-k\sigma_3}e^{i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3} = \mathbb{I} + \mathcal{O}\left(\frac{1}{\zeta}\right), \qquad \text{as } \zeta\to\pm\infty.$$

Then define the functions

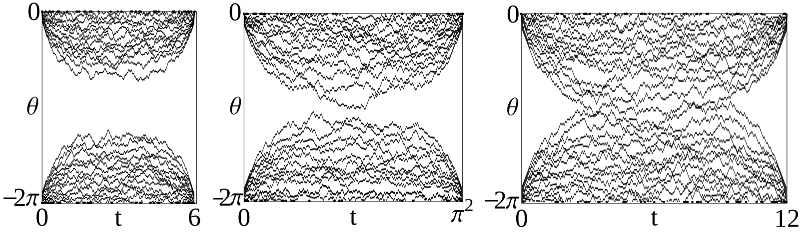
$$f_k(u;s) := \begin{cases} -[\widetilde{\mathbf{L}}_k(u;s)]_{12}, & \operatorname{Im} u > 0, \\ [\widetilde{\mathbf{L}}_k(u;s)]_{11}, & \operatorname{Im} u < 0, \end{cases} \qquad g_k(u;s) := \begin{cases} -[\widetilde{\mathbf{L}}_k(u;s)]_{22}, & \operatorname{Im} u > 0, \\ [\widetilde{\mathbf{L}}_k(u;s)]_{21}, & \operatorname{Im} u < 0. \end{cases}$$

The k-tacnode kernel is then (Buckingham–L. 2017)

$$K_{\mathrm{tac}}^{(k)}(\xi,\eta;\sigma,t) := \frac{1}{2\pi}\int_{\Sigma_T} du \int_{\Sigma_T} dv\, e^{\frac{t}{2}(u^2-v^2)-i(\xi u-\eta v)}\frac{f_k(u;\sigma)g_k(v;\sigma) - g_k(u;\sigma)f_k(v;\sigma)}{2\pi i(u-v)}.$$

Nonintersecting Brownian motions on the circle

These tacnode kernels were obtained as scaling limits of nonintersecting Brownian motions on the unit circle. Let $\mathbb{T} = \{e^{i\theta} | \theta \in \mathbb{R}\}$ be the unit circle. We consider n Brownian bridges on $\mathbb{T}$ with diffusion parameter $1/\sqrt{n}$ and (possibly) a drift $\mu \geq 0$, conditioned to begin at $\theta = 0$ at time $t = 0$ and to end at $\theta = 0$ at time $t = T > 0$.



Denote this process $\text{NIBM}^{\mu}_{0 \to T}$.

The kernel for NIBM on $\mathbb{T}$

Introduce the lattice

$$L_{n,\tau} = \left\{ \frac{k+\tau}{n} \;:\; k \in \mathbb{Z} \right\}.$$

and the monic orthogonal polynomials $p_j(x) = x^j + \ldots$ such that

$$\frac{1}{n} \sum_{x \in L_{n,\tau}} p_j(x) p_k(x) e^{-\frac{nT}{2}(x^2 - 2i\mu x)} = h_k \delta_{jk}.$$

The kernel for NIBM on $\mathbb{T}$

Introduce the lattice

$$L_{n,\tau} = \left\{ \frac{k+\tau}{n} \ : \ k \in \mathbb{Z} \right\}.$$

and the monic orthogonal polynomials $p_j(x) = x^j + \dots$ such that

$$\frac{1}{n} \sum_{x \in L_{n,\tau}} p_j(x) p_k(x) e^{-\frac{nT}{2}(x^2 - 2i\mu x)} = h_k \delta_{jk}.$$

The Christoffel–Darboux kernel is

$$K^{CD}(x,y) = e^{-\frac{nT(x^2+y^2)}{4}} \sum_{k=0}^{n-1} \frac{p_k(x) p_k(y)}{h_k}$$

The kernel for NIBM on $\mathbb{T}$

Introduce the lattice

$$L_{n,\tau} = \left\{ \frac{k+\tau}{n} \ : \ k \in \mathbb{Z} \right\}.$$

and the monic orthogonal polynomials $p_j(x) = x^j + \dots$ such that

$$\frac{1}{n} \sum_{x \in L_{n,\tau}} p_j(x) p_k(x) e^{-\frac{nT}{2}(x^2 - 2i\mu x)} = h_k \delta_{jk}.$$

The Christoffel–Darboux kernel is

$$\begin{aligned} K^{CD}(x,y) &= e^{-\frac{nT(x^2+y^2)}{4}} \sum_{k=0}^{n-1} \frac{p_k(x)p_k(y)}{h_k} \\ &= e^{-\frac{nT(x^2-2i\mu x+y^2-2i\mu y)}{4}} \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{h_{n-1}(x-y)}. \end{aligned}$$

The kernel for NIBM on $\mathbb{T}$

Introduce the lattice

$$L_{n,\tau} = \left\{ \frac{k+\tau}{n} \;:\; k \in \mathbb{Z} \right\}.$$

and the monic orthogonal polynomials $p_j(x) = x^j + \dots$ such that

$$\frac{1}{n} \sum_{x \in L_{n,\tau}} p_j(x) p_k(x) e^{-\frac{nT}{2}(x^2 - 2i\mu x)} = h_k \delta_{jk}.$$

The Christoffel–Darboux kernel is

$$\begin{aligned} K^{CD}(x,y) &= e^{-\frac{nT(x^2+y^2)}{4}} \sum_{k=0}^{n-1} \frac{p_k(x)p_k(y)}{h_k} \\ &= e^{-\frac{nT(x^2-2i\mu x+y^2-2i\mu y)}{4}} \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{h_{n-1}(x-y)}. \end{aligned}$$

The τ-deformed kernel for $\mathrm{NIBM}^{\mu}_{0\to T}$ is

$$K_n(\varphi, \theta; t) = \frac{1}{2\pi n} \sum_{x,y \in L_{n,\tau}} K^{CD}(x,y) e^{\frac{-n(T/2-t)x^2}{2}} e^{\frac{n(T/2-t)y^2}{2}} e^{in(x\varphi - y\theta)}.$$

The kernel for NIBM on $\mathbb{T}$

Introduce the lattice

$$L_{n,\tau} = \left\{ \frac{k+\tau}{n} \ : \ k \in \mathbb{Z} \right\}.$$

and the monic orthogonal polynomials $p_j(x) = x^j + \dots$ such that

$$\frac{1}{n} \sum_{x \in L_{n,\tau}} p_j(x) p_k(x) e^{-\frac{nT}{2}(x^2 - 2i\mu x)} = h_k \delta_{jk}.$$

The Christoffel–Darboux kernel is

$$\begin{aligned} K^{CD}(x,y) &= e^{-\frac{nT(x^2+y^2)}{4}} \sum_{k=0}^{n-1} \frac{p_k(x)p_k(y)}{h_k} \\ &= e^{-\frac{nT(x^2-2i\mu x+y^2-2i\mu y)}{4}} \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{h_{n-1}(x-y)}. \end{aligned}$$

The τ-deformed kernel for $\mathrm{NIBM}^{\mu}_{0\to T}$ is

$$K_n(\varphi,\theta;t) = \frac{1}{2\pi n} \sum_{x,y \in L_{n,\tau}} K^{CD}(x,y) e^{\frac{-n(T/2-t)x^2}{2}} e^{\frac{n(T/2-t)y^2}{2}} e^{in(x\varphi - y\theta)}.$$

It is only a correlation kernel for $\tau = 0$ (n odd) or $\tau = 1/2$ (n even).

Information on the winding number

Let $\mathcal{W}_n(T,\mu)$ be the random variable counting the total winding number of particles in the process $\text{NIBM}^{\mu}_{0\to T}$. It has the probability mass function:

$$\mathcal{P}(\mathcal{W}_n(T,\mu)=\omega)=e^{2\pi i\omega\epsilon(n)}\int_0^1 \frac{\mathcal{H}_n(T,\mu,\tau)}{\mathcal{H}_n(T,\mu,\epsilon(n))}e^{-2\pi i\omega\tau}d\tau,$$

where $\epsilon(n)=0$ if n is odd, $\epsilon(n)=1/2$ if n is even, and

$$\mathcal{H}_n(T,\mu,\tau):=\prod_{j=0}^{n-1}h_j(T,\mu,\tau),\quad \frac{\partial}{\partial\tau}\mathcal{H}_n(T,\mu,\tau)=inT\mu+Tc_{n,n-1},$$

where

$$p_n(z)=z^n+c_{n,n-1}z^{n-1}+\dots$$

To understand asymptotic distribution of winding numbers, need asymptotics of $c_{n,n-1}$ for large n.

Information on the winding number

Let $\mathcal{W}_n(T,\mu)$ be the random variable counting the total winding number of particles in the process $\mathrm{NIBM}^{\mu}_{0\to T}$. It has the probability mass function:

$$\mathcal{P}(\mathcal{W}_n(T,\mu)=\omega) = e^{2\pi i\omega\epsilon(n)} \int_0^1 \frac{\mathcal{H}_n(T,\mu,\tau)}{\mathcal{H}_n(T,\mu,\epsilon(n))} e^{-2\pi i\omega\tau}\, d\tau,$$

where $\epsilon(n)=0$ if n is odd, $\epsilon(n)=1/2$ if n is even, and

$$\mathcal{H}_n(T,\mu,\tau) := \prod_{j=0}^{n-1} h_j(T,\mu,\tau), \quad \frac{\partial}{\partial\tau}\mathcal{H}_n(T,\mu,\tau) = inT\mu + Tc_{n,n-1},$$

where

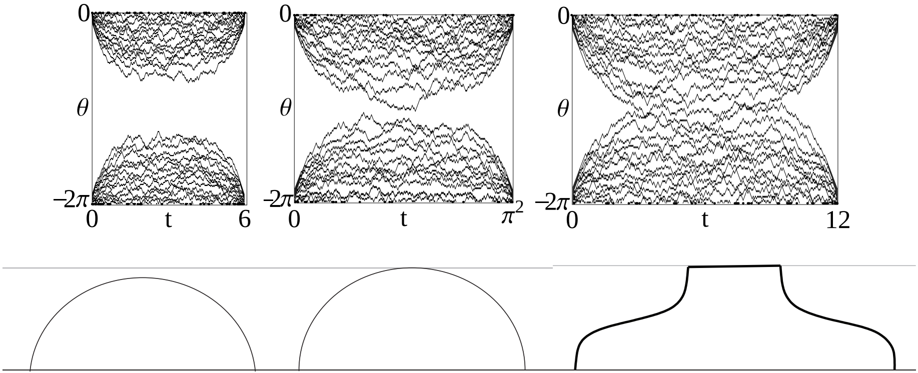
$$p_n(z) = z^n + c_{n,n-1}z^{n-1} + \dots$$

To understand asymptotic distribution of winding numbers, need asymptotics of $c_{n,n-1}$ for large n.

To understand local scaling limits as $n\to\infty$,

1. Replace double sum in kernel with double contour integral.
2. Find asymptotic expressions for the orthogonal polynomials (Riemann–Hilbert analysis).
3. Insert asymptotics into the integral and do classical steepest descent analysis.

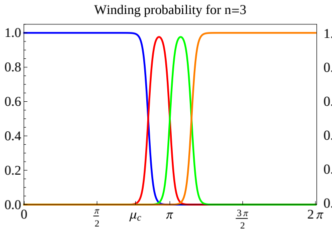
Basic picture for $\mu = 0$.



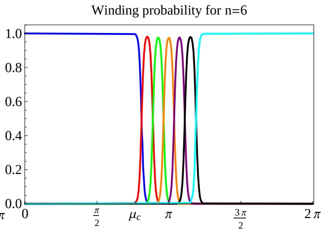
Three phases:

- subcritical: $0 < T < \pi^2$, probability of nonzero winding number is exponentially small in n.
- critical: $T = \pi^2(1 - 2^{-2/3}\sigma n^{-2/3})$ for $\sigma \in \mathbb{R}$, probability of nonzero winding number is algebraically small in n, with coefficients given by HM solution to PII $u_{\mathrm{HM}}(\sigma)$.
- supercritical: $T > \pi^2$, distribution of winding numbers approaches a discrete normal distribution.

Basic picture for $\mu > 0$, $0 < T < \pi^2$.

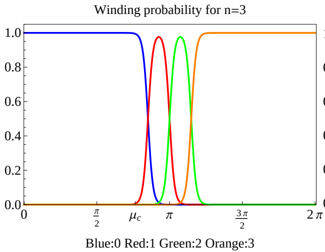


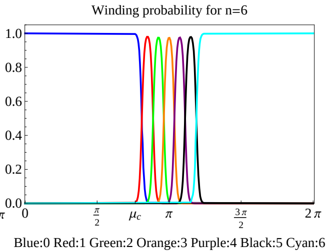
Blue:0 Red:1 Green:2 Orange:3



Blue:0 Red:1 Green:2 Orange:3 Purple:4 Black:5 Cyan:6

Basic picture for $\mu > 0$, $0 < T < \pi^2$.

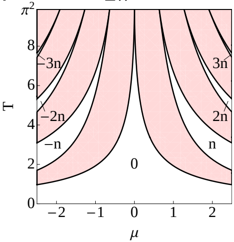




We can describe μ_c explicitly:

$$\mu_c(T) := \frac{\sqrt{\pi^2 - T}}{T} - \frac{\log T}{2\pi} + \frac{\log(\pi - \sqrt{\pi^2 - T})}{\pi},$$

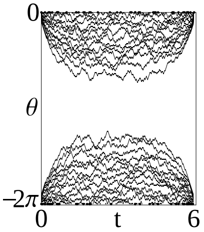
and conjecture the picture



Rigorous results for $0 < T < \pi^2$

For $0 < T < \pi^2$ and $|\mu| < \mu_c(T)$, we have

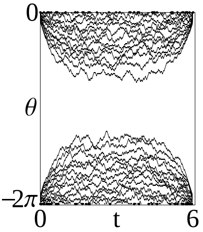
$$\mathbb{P}(\mathcal{W}_n(T, \mu) = 0) = 1 + \mathcal{O}(e^{-cn}), \qquad c > 0.$$



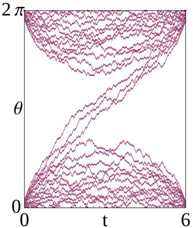
Rigorous results for $0 < T < \pi^2$

For $0 < T < \pi^2$ and $|\mu| < \mu_c(T)$, we have

$$\mathbb{P}(\mathcal{W}_n(T,\mu) = 0) = 1 + \mathcal{O}(e^{-cn}), \qquad c > 0.$$



For μ just a bit bigger than μ_c, we see the winding number start to increase:



Rigorous results for $0 < T < \pi^2$

Define

$$F_j := \frac{j!}{2^{j+1}\sqrt{\pi}} \left(\frac{T}{(2\pi)^{3/2}(\pi^2 - T)^{1/4}} \right)^{2j+1} \frac{e^{2\pi n(\mu - \mu_c(T))}}{n^{j+(1/2)}}$$

Fix a return time $T \in (0, \pi^2)$, a positive integer k, and μ satisfying

$$\mu_c + \left(k - \frac{1}{2}\right)\frac{\log n}{2\pi n} < \mu \leq \mu_c + \left(k + \frac{1}{2}\right)\frac{\log n}{2\pi n}.$$

Then

$$\mathbb{P}(\mathcal{W}_n(T,\mu) = \omega) = \begin{cases} \dfrac{F_{k-1}^{-1}}{1 + F_k + F_{k-1}^{-1}} + \mathcal{O}\left(\dfrac{1}{n}\right), & \omega = k-1, \\ \dfrac{1}{1 + F_k + F_{k-1}^{-1}} + \mathcal{O}\left(\dfrac{1}{n}\right), & \omega = k, \\ \dfrac{F_k}{1 + F_k + F_{k-1}^{-1}} + \mathcal{O}\left(\dfrac{1}{n}\right), & \omega = k+1, \\ \mathcal{O}\left(\dfrac{1}{n}\right), & \text{otherwise.} \end{cases}$$

Here $F_{k-1} = o(1)$ except at $\mu = \mu_c + (k - \frac{1}{2})\frac{\log n}{2\pi n}$, and $F_k = o(1)$ except at $\mu = \mu_c + (k + \frac{1}{2})\frac{\log n}{2\pi n}$.

Rigorous results for $T \approx \pi^2$

Fix $\sigma \in \mathbb{R}$ and let $T = \pi^2(1 - 2^{-2/3}\sigma n^{-2/3})$. Also fix a non-negative integer k and let

$$\left(k - \frac{1}{2}\right)\frac{\log n}{3\pi n} < \mu \leq \left(k + \frac{1}{2}\right)\frac{\log n}{3\pi n}.$$

Then

$$\mathcal{P}(\mathcal{W}_n(T,\mu) = \omega) = \begin{cases} \dfrac{F_{\mathcal{V}}}{1 + F_{\mathcal{U}} + F_{\mathcal{V}}} + \mathcal{O}\left(\dfrac{1}{n^{2/3}}\right), & \omega = k-1, \\ \dfrac{1}{1 + F_{\mathcal{U}} + F_{\mathcal{V}}} + \mathcal{O}\left(\dfrac{1}{n^{2/3}}\right), & \omega = k, \\ \dfrac{F_{\mathcal{U}}}{1 + F_{\mathcal{U}} + F_{\mathcal{V}}} + \mathcal{O}\left(\dfrac{1}{n^{2/3}}\right), & \omega = k+1, \\ \mathcal{O}\left(\dfrac{1}{n^{2/3}}\right), & \text{otherwise}, \end{cases}$$

where

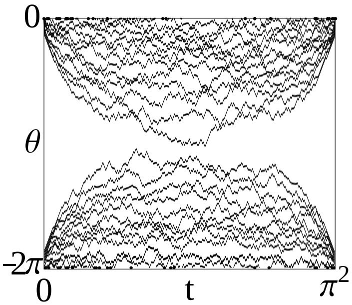
$$F_{\mathcal{U}} \equiv F_{\mathcal{U}}(T,\mu,n) := \frac{ie^{2n\pi\mu}}{2(2n)^{(2k+1)/3}}\mathcal{U}_k(\sigma),$$

$$F_{\mathcal{V}} \equiv F_{\mathcal{V}}(T,\mu,n) := \frac{-i(2n)^{(2k-1)/3}}{2e^{2n\pi\mu}}\mathcal{V}_k(\sigma).$$

Rigorous results for $T \approx \pi^2$

Scale $T = \pi^2(1 - 2^{-2/3}\sigma n^{-2/3})$,

$$t = \frac{T}{2} + \frac{2^{-10/3}\pi^2}{n^{1/3}}\tilde{t}, \quad \varphi = -\pi - \frac{2^{-5/3}\pi}{n^{2/3}}\xi, \quad \theta = -\pi - \frac{2^{-5/3}\pi}{n^{2/3}}\eta, \quad \mu = \frac{k\log(n)}{3\pi n}.$$



Then

$$\lim_{n\to\infty} K_n(\varphi, \theta; t) = K_{\mathrm{tac}}^{(k)}(\xi, \eta; \sigma, \tilde{t}).$$

Orthogonal polynomials with varying complex exponential weights

For comparison, consider the continuous version of our discrete OP's, i.e., let $H_k^\mu(x)$ be the monic polynomial of degree k satisfying

$$\int_{\mathbb{R}} H_k^\mu(x) H_j^\mu(x) e^{-\frac{nT}{2}(x^2-2i\mu x)} dx = h_k^\mu \delta_{jk},$$

For general μ, we can complete the square in the exponent to obtain

$$\int_{\mathbb{R}} H_k^\mu(x) H_j^\mu(x) e^{-\frac{nT}{2}(x-i\mu)^2} dx = e^{nT\mu^2/2} h_k^\mu \delta_{jk},$$

or equivalently

$$\int_{\mathbb{R}-i\mu} H_k^\mu(x+i\mu) H_k^\mu(x+i\mu) e^{-\frac{nT}{2}x^2} dx = e^{nT\mu^2/2} h_k^\mu \delta_{jk}.$$

Using Cauchy's theorem, the contour of integration can be deformed back to the real line, giving the orthogonality condition for $H_k^0(x)$.Thus

$$H_k^\mu(x) \equiv H_k^0(x-i\mu).$$

Orthogonal polynomials with varying complex exponential weights

For comparison, consider the continuous version of our discrete OP's, i.e., let $H_k^\mu(x)$ be the monic polynomial of degree k satisfying

$$\int_{\mathbb{R}} H_k^\mu(x) H_j^\mu(x) e^{-\frac{nT}{2}(x^2 - 2i\mu x)} dx = h_k^\mu \delta_{jk},$$

For general μ, we can complete the square in the exponent to obtain

$$\int_{\mathbb{R}} H_k^\mu(x) H_j^\mu(x) e^{-\frac{nT}{2}(x - i\mu)^2} dx = e^{nT\mu^2/2} h_k^\mu \delta_{jk},$$

or equivalently

$$\int_{\mathbb{R}-i\mu} H_k^\mu(x + i\mu) H_k^\mu(x + i\mu) e^{-\frac{nT}{2}x^2} dx = e^{nT\mu^2/2} h_k^\mu \delta_{jk}.$$

Using Cauchy's theorem, the contour of integration can be deformed back to the real line, giving the orthogonality condition for $H_k^0(x)$.Thus

$$H_k^\mu(x) \equiv H_k^0(x - i\mu).$$

For discrete weights, there is no way to use Cauchy's theorem, so there is no such relation.

Discrete Riemann–Hilbert problem

Discrete RHP: Find the 2×2 matrix-valued function $\mathbf{P}_n(z)$ such that

- $\mathbf{P}_n(z)$ is meromorphic with poles at $L_{n,\tau}$.
- $\operatorname*{Res}_{z=x} \mathbf{P}_n(z)_{j2} = \frac{1}{n} e^{-\frac{nT}{2}(x^2 - 2i\mu x)} \mathbf{P}_n(x)_{j1}, \quad j = 1, 2.$
- $\mathbf{P}_n(z) = \left(\mathbb{I} + \frac{\mathbf{P}_1}{z} + \frac{\mathbf{P}_2}{z^2} + \dots\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad z$ bounded away from $L_{n,\tau}$.

Discrete Riemann–Hilbert problem

Discrete RHP: Find the 2×2 matrix-valued function $\mathbf{P}_n(z)$ such that

- $\mathbf{P}_n(z)$ is meromorphic with poles at $L_{n,\tau}$.
- $\operatorname*{Res}_{z=x} \mathbf{P}_n(z)_{j2} = \frac{1}{n} e^{-\frac{nT}{2}(x^2-2i\mu x)} \mathbf{P}_n(x)_{j1}, \quad j=1,2.$
- $\mathbf{P}_n(z) = \left(\mathbb{I} + \frac{\mathbf{P}_1}{z} + \frac{\mathbf{P}_2}{z^2} + \dots\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad z$ bounded away from $L_{n,\tau}$.

The unique solution to the Riemann-Hilbert Problem is

$$\mathbf{P}_n(z) := \begin{pmatrix} p_n(z) & (Cp_n)(z) \\ (h_{n-1})^{-1} p_{n-1}(z) & (h_{n-1})^{-1} (Cp_{n-1})(z) \end{pmatrix},$$

where the weighted discrete Cauchy transform C is

$$Cf(z) := \frac{1}{n} \sum_{x \in L_{n,\tau}} \frac{f(x) e^{-(nT/2)(x^2-2i\mu x)}}{z-x}.$$

Also:

$$c_{n,n-1} = [\mathbf{P}_1]_{11}.$$

Steepest descent analysis

The steepest descent analysis transforms $\mathbf{P}_n(z)$ into a matrix solving a RHP which can be solved explicitly up to an explicit error.

1. Interpolation of poles – converts discrete RHP to continuous one involving jumps instead of residue conditions. Jumps are on three horizontal lines. One must be above $\mathbb{R}$ and one must be below, with the third lying inbetween.

Steepest descent analysis

The steepest descent analysis transforms $\mathbf{P}_n(z)$ into a matrix solving a RHP which can be solved explicitly up to an explicit error.

1. Interpolation of poles – converts discrete RHP to continuous one involving jumps instead of residue conditions. Jumps are on three horizontal lines. One must be above $\mathbb{R}$ and one must be below, with the third lying inbetween.
2. Introduction of the g-function – normalizes asymptotics and prepares existing jumps for the next step. In the case $|\mu| \leq \mu_c$, we have $g_\mu(z) = g_0(z - i\mu)$.

Steepest descent analysis

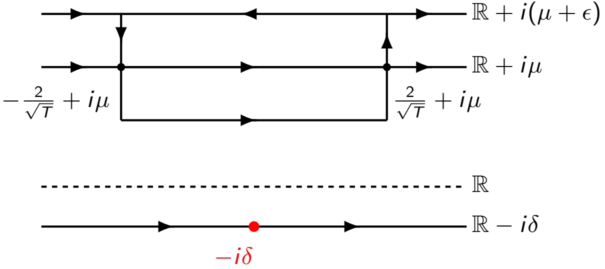
The steepest descent analysis transforms $\mathbf{P}_n(z)$ into a matrix solving a RHP which can be solved explicitly up to an explicit error.

1. Interpolation of poles – converts discrete RHP to continuous one involving jumps instead of residue conditions. Jumps are on three horizontal lines. One must be above $\mathbb{R}$ and one must be below, with the third lying inbetween.
2. Introduction of the g-function – normalizes asymptotics and prepares existing jumps for the next step. In the case $|\mu| \leq \mu_c$, we have $g_\mu(z) = g_0(z - i\mu)$.
3. Opening of the lenses near the cut of the g-function (aka support of the equilibrium measure).

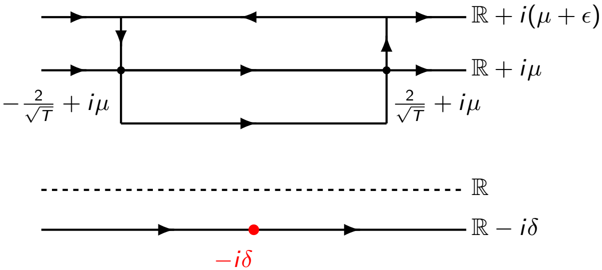
Steepest descent analysis

The steepest descent analysis transforms $\mathbf{P}_n(z)$ into a matrix solving a RHP which can be solved explicitly up to an explicit error.

1. Interpolation of poles – converts discrete RHP to continuous one involving jumps instead of residue conditions. Jumps are on three horizontal lines. One must be above $\mathbb{R}$ and one must be below, with the third lying inbetween.
2. Introduction of the g-function – normalizes asymptotics and prepares existing jumps for the next step. In the case $|\mu| \leq \mu_c$, we have $g_\mu(z) = g_0(z - i\mu)$.
3. Opening of the lenses near the cut of the g-function (aka support of the equilibrium measure).

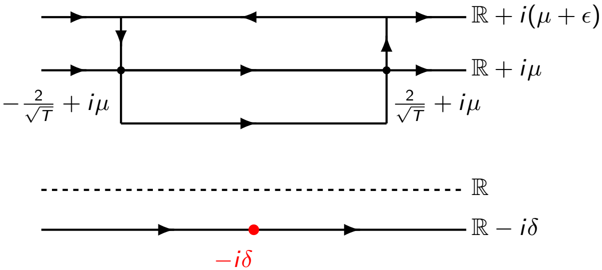


Steepest descent analysis, $0 < T < \pi^2$



Since the g-function is well controlled close to its branch cut, jumps above $\mathbb{R}$ are well controlled. Difficulty is that the jump below $\mathbb{R}$ is far away from the branch cut and it is in general difficult to get information about g-function there.

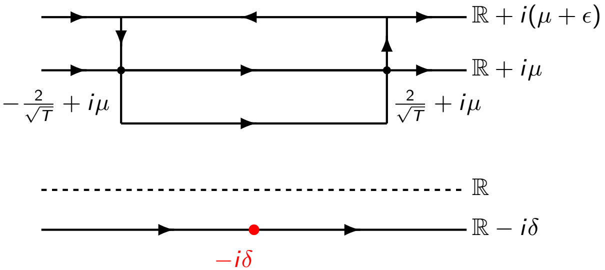
Steepest descent analysis, $0 < T < \pi^2$



Since the g-function is well controlled close to its branch cut, jumps above $\mathbb{R}$ are well controlled. Difficulty is that the jump below $\mathbb{R}$ is far away from the branch cut and it is in general difficult to get information about g-function there.
Since we have an explicit formula for $g_\mu(z)$, we can just check

- For $0 \leq \mu < \mu_c$, jump on $\mathbb{R} - i\delta$ is exponentially close to $\mathbb{I}$.

Steepest descent analysis, $0 < T < \pi^2$

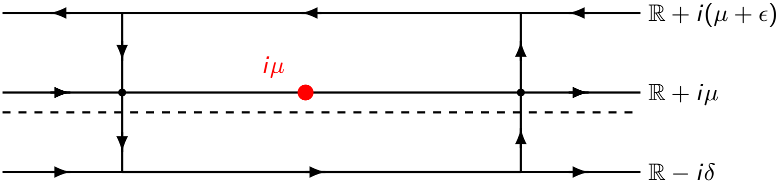


Since the g-function is well controlled close to its branch cut, jumps above $\mathbb{R}$ are well controlled. Difficulty is that the jump below $\mathbb{R}$ is far away from the branch cut and it is in general difficult to get information about g-function there.
Since we have an explicit formula for $g_\mu(z)$, we can just check

- For $0 \leq \mu < \mu_c$, jump on $\mathbb{R} - i\delta$ is exponentially close to $\mathbb{I}$.
- For $0 < T < \pi^2$ and $\mu = \mu_c + \mathcal{O}(\log(n)/n)$, jump is not small at the single point $z = -i\delta$. Local solution is given in terms of Hermite functions. Very similar to "birth of a cut" in random matrix models.

Steepest descent analysis: $T \approx \pi^2$

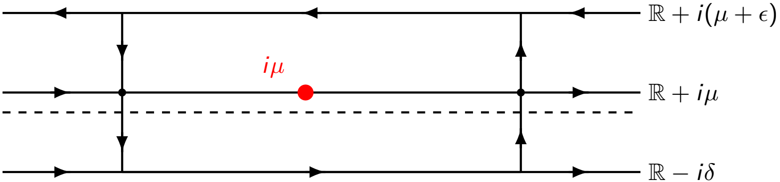
For $T \approx \pi^2$, we have $\mu_c = 0$ and the analysis is different.



For $\mu = \frac{k \log(n)}{3\pi n}$, local solution near $z = i\mu$ is given in terms of solution to Jimbo–Miwa–Garnier Lax pair for PII with parameter $\alpha = k - \pm 1/2$. This gives formula for k-tacnode kernel.

Steepest descent analysis: $T \approx \pi^2$

For $T \approx \pi^2$, we have $\mu_c = 0$ and the analysis is different.

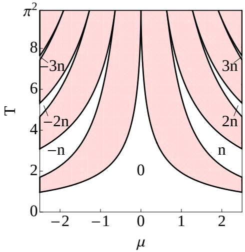


For $\mu = \frac{k \log(n)}{3\pi n}$, local solution near $z = i\mu$ is given in terms of solution to Jimbo–Miwa–Garnier Lax pair for PII with parameter $\alpha = k - \pm 1/2$. This gives formula for k-tacnode kernel.

In special case $k = 0$, it reduces to Flashka–Newell Lax pair, giving formula for usual tacnode kernel.

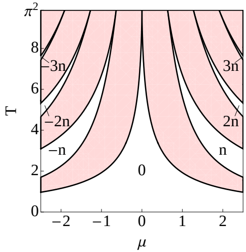
Further directions

- What happens for $\mu > \mu_c$? Two-cut RHP, elliptic functions, etc. Can we prove this picture?



Further directions

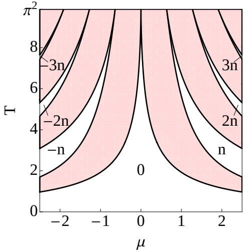
- What happens for $\mu > \mu_c$? Two-cut RHP, elliptic functions, etc. Can we prove this picture?



- What about supercritical phase $T > \pi^2$? Winding numbers given by shifted discrete normal distribution? What about orthogonal polynomials?

Further directions

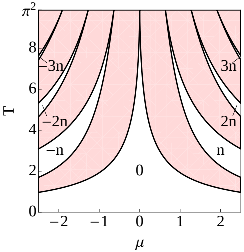
- What happens for $\mu > \mu_c$? Two-cut RHP, elliptic functions, etc. Can we prove this picture?



- What about supercritical phase $T > \pi^2$? Winding numbers given by shifted discrete normal distribution? What about orthogonal polynomials?
- Asymptotics of discrete OP's with weight $w(x) = e^{-n(V(x)+i\mu x)}$?

Further directions

- What happens for $\mu > \mu_c$? Two-cut RHP, elliptic functions, etc. Can we prove this picture?



- What about supercritical phase $T > \pi^2$? Winding numbers given by shifted discrete normal distribution? What about orthogonal polynomials?
- Asymptotics of discrete OP's with weight $w(x) = e^{-n(V(x)+i\mu x)}$?
- More general discrete OP's with complex weights?

References

- K. Liechty and D. Wang, Nonintersecting Brownian motions on the unit circle, *Ann. Probab.* **44**, 1134–1211 (2016).
- R. Buckingham and K. Liechty, Nonintersecting Brownian bridges on the unit circle with drift, arXiv:1707.07211 (2017).
- R. Buckingham and K. Liechty, The k-tacnode process, arXiv:1709......

Thanks

Thanks!!