Andrei Kapaev and the asymptotic analysis of Painlevé transcendents.

Alexander Its

Indiana University-Purdue University
Indianapolis

University of Michigan, Ann Arbor, August 26, 2017

This talk is a tribute to the memory of

Andrei Anatolievich Kapaev

(1962 - 2016)

CONNECTION FORMULAE FOR THE PAINLEVÉ EQUATIONS

THE BEGINNING

1977 - 1980

The first connection formulae

- B. M. McCoy, C. A. Tracy, T. T. Wu: special family of the third Painlevé transcendents related to the Ising model
- M. J. Ablowitz, H. Segur: special family of the second Painlevé transcendents related to the KdV equation

The introduction of the Isomonodrtomy Method

- H. Flaschka, A. C. Newell: Painlevé II, III
- M. Jimbo, T. Miwa, K. Ueno: Painlevé I - VI

ANDREI'S EARLY WORKS

The Isomonodrtomy approach to the asymptotic analysis

1986 - 1998

PAINLEVÉ II

ASYMPTOTICS ON THE REAL LINE

1986

The second Pianlevé equation:

$$u_{xx} = xu + 2u^3$$

Isomonodromic deformations of (Flaschka-Newell):

$$\frac{d\Psi}{d\lambda} = A(\lambda)\Phi, \quad A(\lambda) = -4i\lambda^2\sigma_3 + \lambda A_1 + A_0$$

$$A_1 = \begin{pmatrix} 0 & 4iu \\ -4iu & 0 \end{pmatrix} \quad A_0 = \begin{pmatrix} -ix - 2iu^2 & -2u_x \\ -2u_x & ix + 2iu^2 \end{pmatrix}$$

Canonical Solutions

$$\Psi_k(\lambda) \simeq \left(I + \sum_{j=1}^{\infty} \frac{m_j}{\lambda^j} \right) e^{-\frac{4i}{3}\lambda^3\sigma_3 - ix\lambda\sigma_3}$$

as

$$\lambda \to \infty, \quad \lambda \in \Omega_k, \quad k = 1, \ldots, 7$$

where

$$\Omega_k = \left\{ \lambda : (k-2)\frac{\pi}{3} < \arg\lambda < k\frac{\pi}{3} \right\}$$

Monodromy Data

Stokes matrices, S_k:

$$\Psi_{k+1} = \Psi_k S_k, \quad k = 1, \dots, 6$$

$$S_{2l} = \begin{pmatrix} 1 & s_{2l} \\ 0 & 1 \end{pmatrix}, \quad S_{2l+1} = \begin{pmatrix} 1 & 0 \\ s_{2l+1} & 1 \end{pmatrix}$$

$$S_1 \dots S_6 = I, \quad S_{k+3} = \sigma_2 S_k \sigma_2$$

$$\Downarrow$$

$$s_4 = -s_1, \quad s_5 = -s_2, \quad s_6 = -s_3,$$

$$s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0$$

$$\mathcal{M} = \left\{ s = (s_1, s_2, s_3) \in \mathbb{C}^3 : s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0 \right\}$$

IM in short:

- $s \equiv s(u, u_x, x)$ are the first integrals of PII (Direct Monodromy Problem)
- The set of all solutions of PII $\simeq \mathcal{M}$.
- $u(x) \equiv u(x; s)$ (Inverse Monodromy Problem)

DMP Approach to the Connection Problem

Theorem 1 Let u and u_x be purely imaginary functions of x defined on $(-\infty, x_-]$ satisfying the following conditions:

a)

$$u(x) = O((-x)^{-1/4}), \quad u_x(x) = O((-x)^{1/4}), \quad x \to -\infty. \tag{1}$$

b) There exist constants $c > 0$ and $X_- \leq x_-$, such that

$$|v(x)| \geq c \qquad \forall x \in (-\infty, X_-], \tag{2}$$

where

$$v = \frac{i}{\sqrt{2}} e^{i\pi/4}[(-x)^{1/4} u + i(-x)^{-1/4} u_x].$$

Then the associated monodromy data $s = s(u, u_x, x)$, satisfy the following asymptotic equation

$$s_1 = \frac{i}{v}\frac{\sqrt{2\pi}}{\Gamma(i|v|^2)}\exp\Big\{\frac{2}{3}i(-x)^{3/2} + i\frac{3}{2}|v|^2\log(-x)$$

$$+i|v|^2\log 8 + \frac{\pi}{2}|v|^2\Big\} + o(1), \quad x \to -\infty,$$

$$v = \frac{i}{\sqrt{2}}e^{i\pi/4}[(-x)^{1/4}u + i(-x)^{-1/4}u_x].$$

$$s_3 = -\bar{s}_1, \quad s_2 = \frac{s_1 - \bar{s}_1}{1 + |s_1|^2}.$$

Here, $\Gamma(z)$ is the Euler gamma-function.

Theorem 2 Let u, u_x be purely imaginary functions of x defined on $[x_+, \infty)$ and satisfying the following conditions:

a)

$$u(x) = i\sqrt{\frac{x}{2}} + u_0(x), \;\; u_0 = O(x^{-1/4}), \quad x \to +\infty, \qquad (3)$$

$$u_x(x) = \frac{i}{2}\sqrt{\frac{1}{2x}} + w_0(x), \;\; w_0 = O(x^{1/4}), \quad x \to +\infty.$$

b) There exist constants $c > 0$ and $X_+ \geq x_+$, such that

$$|v_0(x)| \geq c, \qquad \forall x \in [X_+, \infty), \qquad (4)$$

where

$$v_0 = e^{i\pi/4}\left[(2x)^{1/4}u_0 + i(2x)^{-1/4}w_0\right].$$

Then the associated monodromy data $s_1 = s_1(u, u_x, x)$, satisfy the following asymptotic equations

$$\frac{\bar{s}_1 - s_1}{1 + |s_1|^2} = ie^{-\pi|v_0|^2} + o(1), \qquad x \to +\infty,$$

$$\frac{1 + s_1^2}{1 + |s_1|^2} = -\frac{i}{v_0} \frac{\sqrt{2\pi}}{\Gamma(i|v_0|^2)} \exp\Big\{ -\frac{2i\sqrt{2}}{3} x^{3/2}$$

$$+ i\frac{3}{2}|v_0|^2 \log x + i\frac{7}{2}|v_0|^2 \log 2 - \frac{\pi}{2}|v_0|^2 \Big\} + o(1),$$

$$x \to +\infty.$$

$$s_1 \equiv \text{const} \quad \Longrightarrow \text{connection formulae :}$$

- $x \to -\infty$:

$$u(x) = id(-x)^{-1/4} \sin\left\{\frac{2}{3}(-x)^{3/2} + \frac{3}{4}d^2 \log(-x) + \varphi\right\}$$

$$+0((-x)^{-1}),$$

where

$$d^2 = \frac{1}{\pi}\log(1+|s_1|^2), \quad d > 0,$$

$$\varphi = \frac{3}{2}d^2 \log 2 - \frac{\pi}{4} - \arg\Gamma\left(i\frac{d^2}{2}\right) - \arg s_1 .$$

- $x \to +\infty$:

1) If $\Im s_1 \neq 0$, then

$$u(x) = i\sigma\sqrt{\frac{x}{2}} + i\sigma(2x)^{-1/4}\rho\cos\left\{\frac{2\sqrt{2}}{3}x^{3/2} - \frac{3}{2}\rho^2\log x + \theta\right\}$$

$$+O(x^{-1}),$$

where

$$\rho^2 = \frac{1}{\pi}\log\frac{1+|s_1|^2}{2|\Im s_1|}, \quad \rho > 0, \quad \sigma = -\mathrm{sign}\,\Im s_1,$$

$$\theta = -\frac{3\pi}{4} - \frac{7}{2}\rho^2\log 2 + \arg\Gamma(i\rho^2) + \arg(1+s_1^2).$$

2) If $\Im s_1 = 0$, then

$$u(x) = \frac{is_1}{2\sqrt{\pi}}x^{-1/4}e^{-\frac{2}{3}x^{3/2}}\left(1 + O(x^{-3/4})\right),$$

Sketch of the proof of Theorems 1. Complex WKB method

Step 1. Rescaling:

$$\lambda \to z = (-x)^{-1/2}\lambda$$

$$\frac{d\Psi}{d\lambda} = A(\lambda)\Psi \quad \to \quad \frac{d\Psi}{dz} = tA_0(z;t)\Psi,$$

$$t = (-x)^{3/2}, \quad A_0(z;t) = -(4iz^2 - i)\sigma_3 + O\left(\frac{1}{\sqrt{t}}\right).$$

Step 2. WKB-ansatz:

$$\Psi^{\mathrm{WKB}}(z) = T(z)\exp\left\{t\int \Lambda(z)dz\right\}$$

$$T\Lambda T^{-1} = A_0, \quad \Lambda(z) = -\mu(z)\sigma_3$$

$$\mu(z) = \sqrt{-\det A_0(z)} = 4i\left(z^2 - \frac{1}{4}\right) + O\left(\frac{1}{t\left(z^2 - \frac{1}{4}\right)}\right)$$

$$T(z) = I + O\left(\frac{z}{\sqrt{t}\left(z^2 - \frac{1}{4}\right)}\right)$$

Step 3. Turning points and Stokes lines:

$$\mu(z) = 0 \quad \rightarrow \quad z = z_{1,2,3,4},$$

$$z_1 \rightarrow \frac{1}{2} \leftarrow z_3, \quad z_2 \rightarrow -\frac{1}{2} \leftarrow z_4$$

$$\Re \int_{z_{1,2,3,4}}^{z} \mu(z')dz' = 0 \quad \rightarrow \quad \mathbb{R} \cup \mathcal{L},$$

$$\mathcal{L} = \left\{ z = \xi + i\eta : \eta^2 = 3\left(\xi^2 - \frac{1}{4}\right) \right\}$$

Step 4. Canonical WKB solutions:

$$\Psi_k^{\mathrm{WKB}}(z) = \left[I + O\left(\frac{z}{\sqrt{t}\,(z^2 - \frac{1}{4})} \right) \right] \exp\left\{ -t \int_{\frac{1}{2}}^{z} \mu(z')dz' \right\},$$

$$t \to \infty, \left| \sqrt{t}\left(z^2 - \frac{1}{4} \right) \right| \to \infty, \quad z \in \Omega_k^{\mathrm{WKB}},$$

$$k = 1, 2, \quad \Omega_k^{\mathrm{WKB}} \cap \Omega_k \neq \varnothing$$

Step 5. Turning point solution:

$$z \in D_{+}^{\epsilon} = \left\{ z : \left| z - \frac{1}{2} \right| \leq t^{-1/2+\epsilon} \right\} :$$

$$\frac{d\Psi}{dz} = tA_0(z;t)\Psi \quad \sim \quad \frac{d\Psi}{dz} = t\left[-4i\left(z - \frac{1}{2} \right) + 0\left(t^{-1/2} \right) \right] \Psi$$

$$\Downarrow$$

$$\Psi_+(z) = \big(I + o(1)\big)\Psi_0(z), \quad z \to \infty,$$

$$z \in D_+^\epsilon, \quad \arg\left(z - \frac{1}{2}\right) = \frac{\pi}{2}, \quad k = 1, 2,$$

where

$$\Psi_0(z) = \Big\{D_\nu(\zeta)\Big\},$$

$$\zeta = 2\sqrt{2t}\left(z - \frac{1}{2}\right)e^{-\pi/4}, \quad \nu = -i|v|^2 - 1,$$

$$\left(v = \frac{i}{\sqrt{2}}e^{i\pi/4}[(-x)^{1/4}u + i(-x)^{-1/4}u_x]\right)$$

$$\Psi_0(z) = \begin{pmatrix} D_{-1+\nu}(i\zeta) & D_{\nu}(\zeta) \\ D^{\circ}_{-1+\nu}(i\zeta) & D^{\circ}_{\nu}(\zeta) \end{pmatrix},$$

where "$\circ$" means the transform

$$f(z) \mapsto f^{\circ}(\zeta) = \frac{1}{v}\left(f_{\zeta} - \frac{\zeta}{2} f\right),$$

Step 6. Matching and asymptotic evaluation of S_1.

$$\Psi_k(z) = \Psi_k^{\mathrm{WKB}}(z) C_k, \quad C_k \equiv C_k(t), \quad k = 1, 2$$

$$\Psi_k^{\mathrm{WKB}}(z) = \Psi_+(z) N_k, \quad N_k \equiv N_k(t), \quad k = 1, 2$$

$$S_1 = C_1^{-1} N_1^{-1} N_2 C_2.$$

Large t asymptotics of $C_k(t)$ follows from the matching $\Psi_k^{\mathrm{WKB}}(z)$ and $\Psi_k(z)$ for large z. In fact, one has,

$$C_k = \lim_{z\to\infty} \exp\left\{ t\int_{1/2}^{z} \mu(z')dz'\sigma_3 - t\left(\frac{4i}{3}z^3 - iz\right)\sigma_3 \right\}$$

Large t asymptotics of $N_k(t)$ follows from the matching $\Psi_k^{\mathrm{WKB}}(z)$ and $\Psi_+(z)$ in the domain,

$$\frac{1}{2}t^{-1/2+\epsilon} < \left|z-\frac{1}{2}\right| < t^{-1/2+\epsilon}, \quad \arg\left(z-\frac{1}{2}\right) = \frac{\pi}{2}(k-1), \quad k = 1,2$$

Crucial point is that the Stokes matrices for the model solution $\Psi_0(z)$ are explicitly known.

Kapaev's approach to justification via the IMP

$$S_k = \Psi_k^{-1}(\lambda)\Psi_k(\lambda) \quad \to \quad G_Y(\lambda) = Y_-^{-1}(\lambda)Y_+(\lambda), \quad \lambda \in \Sigma$$

where

$$\Sigma = \cup_{k=1}^{6}\Sigma_k, \quad \Sigma_k = \left\{\lambda : \arg\lambda = \frac{2k-1}{6}\pi\right\}, \quad k = 1, ..., 6.$$

$$G_Y(\lambda) = e^{-\frac{4i}{3}\lambda^3\sigma_3 - ix\lambda\sigma_3} S_k e^{\frac{4i}{3}\lambda^3\sigma_3 + ix\lambda\sigma_3}, \quad \lambda \in \Sigma_k,$$

$$Y(\lambda) \to I, \quad \lambda \to \infty$$

$$\left\{\Psi^{\mathrm{WKB}}(z), \quad \Psi_{\pm}(z)\right\} \quad \to \quad Y_{\mathrm{as}}(\lambda):$$

$$R(\lambda) := Y_{\mathrm{as}}^{-1}(\lambda)Y(\lambda), \quad G_R(\lambda) = I + o(1), \quad x \to -\infty.$$

ASYMPTOTICS ON THE COMPLEX PLANE

1991, 1992

Boutroux equations:

$$\begin{cases} \Re\Big[\Big(\frac{e^{i\phi}}{1+\varkappa^2}\Big)^{3/2}\big(-2\varkappa^2 K'(\kappa) + (1+\varkappa^2)E'(\kappa)\big)\Big] = 0, \\ \Im\Big[\Big(\frac{e^{i\phi}}{1+\varkappa^2}\Big)^{3/2}\big((\varkappa^2-1)K(\kappa) + (1+\varkappa^2)E(\kappa)\big)\Big] = 0, \end{cases}$$

with conditions,

$$\phi \to 0 \Rightarrow \varkappa \to 1, \quad \phi \to \frac{\pi}{3} \Rightarrow \varkappa \to 0, \quad -\frac{\pi}{2} < \arg \varkappa < 0,$$

define uniquely the smooth function

$$\kappa = \kappa(\phi), \quad \phi \in \left(0, \frac{\pi}{3}\right).$$

Here $E(\varkappa)$ and $K(\varkappa)$ are standard complete elliptic integrals:

$$K = K(\varkappa) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-\varkappa^2 z^2)}}, \quad K' = K(\varkappa'),$$

$$E = E(\varkappa) = \int_0^1 \sqrt{\frac{1-\varkappa^2 z^2}{1-z^2}}\, dz, \quad E' = E(\varkappa'),$$

$$\varkappa' = \sqrt{1-\varkappa^2}, \quad 0 < \arg \varkappa' < \frac{\pi}{2}, \quad -\frac{\pi}{2} < \arg(1+\varkappa^2) < 0.$$

Theorem 3 Let

$$s_2 \neq 0, \quad 1 + s_2 s_3 \neq 0.$$

Then

$$u(x;s) = -\frac{i\varkappa}{\sqrt{1+\varkappa^2}}\, x^{1/2} \mathrm{sn}\left(\frac{2i}{3}\frac{x^{3/2}}{\sqrt{1+\varkappa^2}} + \frac{2iK}{\pi}\ln(s_2)\right.$$

$$\left. -\frac{K'}{\pi}\ln(1+s_2 s_3) + O\big(x^{-3/2}\big)\big|\varkappa\right),$$

$$x \to \infty, \quad \arg x = \phi + O\big(x^{-3/2}\big), \quad \phi \in \left(0, \frac{\pi}{3}\right),$$

This asymptotics is uniform on the cheese-type domain

$$\mathcal{D} = \left\{x \in \mathbb{C}: \quad |\arg x - \varphi| < C|x|^{-3/2}\right\} \cap D_{\varepsilon},$$

where D_{ε} is the complement to the union of the ε-neighborhoods of all the poles of the above elliptic function.

Theorem 4 Let

$$s_3 \neq 0, \quad 1 - s_1 s_3 \neq 0.$$

Then

$$u(x;s) = -\frac{i\varkappa}{\sqrt{1+\varkappa^2}}\, x^{1/2} \mathrm{sn}\left(\frac{2i}{3}\,\frac{x^{3/2}}{\sqrt{1+\varkappa^2}} - \frac{2iK}{\pi}\ln(s_3)\right.$$

$$\left. + \frac{K'}{\pi}\ln(1 - s_1 s_3) + O\big(x^{-3/2}\big)\Big|\varkappa\right),$$

$$x \to \infty, \quad \arg x = \phi + O\big(x^{-3/2}\big), \quad \phi \in \left(\frac{2\pi}{3}, \pi\right),$$

$$\left(\phi \to \frac{2\pi}{3} \Rightarrow \varkappa \to 1, \quad \phi \to \pi \Rightarrow \varkappa \to 0, \quad -\frac{\pi}{2} < \arg\varkappa < 0\right)$$

This asymptotics is uniform on the cheese-type domain

$$\mathcal{D} = \left\{ x \in \mathbb{C} : \quad |\arg x - \varphi| < C|x|^{-3/2} \right\} \cap D_{\varepsilon},$$

where D_{ε} is the complement to the union of the ε-neighborhoods of all the poles of the above elliptic function.

Other sectors:

$$\arg x \in \left(\frac{\pi}{3}, \frac{2\pi}{3}\right) \quad \Rightarrow \quad \frac{2iK}{\pi}\ln s_3 - \frac{K'}{\pi}\ln(1+s_2 s_3),$$

$$\arg x \in \left(\pi, \frac{4\pi}{3}\right) \quad \Rightarrow \quad -\frac{2iK}{\pi}\ln s_1 + \frac{K'}{\pi}\ln(1-s_1 s_2),$$

$$\arg x \in \left(\frac{4\pi}{3}, \frac{7\pi}{3}\right) \quad \Rightarrow \quad -\frac{2iK}{\pi}\ln s_1 + \frac{K'}{\pi}\ln(1+s_1 s_2),$$

$$\arg x \in \left(\frac{7\pi}{3}, 2\pi\right) \quad \Rightarrow \quad \frac{2iK}{\pi}\ln s_2 - \frac{K'}{\pi}\ln(1+s_1 s_2),$$

SINGULAR SOLUTIONS ON THE REAL LINE

1992

Theorem 5. The real-valued solutions of PII equation have the following asymptotics as $x \to +\infty$:

• If $s_2 \neq 0$, then

$$u(x;s) = \sigma\sqrt{\frac{x}{2}}\cot\Big(\frac{\sqrt{2}}{3}x^{3/2} + \frac{\gamma}{2}\ln(8\sqrt{2}\,x^{3/2}) + \chi\Big) + O(x^{-1}),$$

where

$$\sigma = \operatorname{sign}(s_2) = \pm 1, \quad \gamma = \frac{1}{\pi}\ln(\sigma s_2),$$

$$\chi = -\frac{1}{2}\arg\Gamma\Big(\frac{1}{2} + i\gamma\Big) - \frac{1}{2}\arg(1 + s_2 s_3) + \frac{\pi}{2} + O(\tau^{-1/2})$$

- If $s_2 = 0$, then

$$u(x;s) = a x^{-1/4} e^{-\frac{2}{3}x^{3/2}} \left(1 + O\left(x^{-3/2}\right)\right),$$

where

$$a = -\frac{\Im s_1}{2\sqrt{\pi}}.$$

Theoem 6. The real-valued solutions of PII equation have the following asymptotics as $x \to -\infty$:

• If $|s_1| < 1$, then

$$u(x;s) = (-x)^{-1/4}\sqrt{-2\beta}\cos\Big(\frac{2}{3}(-x)^{3/2} + \beta\ln\big(8(-x)^{3/2}\big) + \phi\Big)$$

$$+O\big((-x)^{-7/10}\big),$$

where

$$\beta = \frac{1}{2\pi}\ln\big(1 - |s_1|^2\big) < 0, \quad \phi = -\frac{\pi}{4} - \arg\Gamma(i\beta) - \arg s_1.$$

- If $|s_1| > 1$, then

$$u(x;s) = \frac{\sqrt{-x}}{\sin\left(\frac{2}{3}(-x)^{3/2} + \beta\ln\left(8(-x)^{3/2}\right) + \phi\right) + O\left((-x)^{-1/5}\right)},$$

$$\beta = \frac{1}{2\pi}\ln\left(|s_1|^2 - 1\right), \phi = -\arg\Gamma\left(\frac{1}{2} + i\beta\right) - \arg s_1.$$

- If $|s_1| = 1$, i.e., $s_1 = i\epsilon$, then

$$u(x;s) = u_-(x) + b(-x)^{-1/4} e^{-\frac{2\sqrt{2}}{3}(-x)^{3/2}} \left(1 + O\left(x^{-3/2}\right)\right),$$

where

$$u_-(x) = -\epsilon\sqrt{-\frac{x}{2}} + O(x^{-5/2}), \quad b = -\frac{s_2}{2^{7/4}\sqrt{\pi}}.$$

TRI - TRONQUÉE SOLUTIONS

1989, 2004

The First Painlevé equation:

$$u_{xx} = 6u^2 + x,$$

The formal solution,

$$u_f(x) = \sqrt{-\frac{x}{6}} \sum_{k=0}^{\infty} a_k (-x)^{-\frac{5}{2}k},$$

$$a_0 = 1,$$

$$a_{k+1} = \frac{25k^2 - 1}{8\sqrt{6}} a_k - \frac{1}{2} \sum_{m=1}^{k} a_m a_{k+1-m}.$$

$$''-'' \equiv e^{-\pi i}$$

Tri-tronquée solutions,

$$u_0(x) \simeq u_f(x),$$

$$x \to \infty, \quad -\frac{\pi}{5} < \arg x < \frac{7\pi}{5}.$$

$$u_1(x) \equiv \overline{u_0(\bar{x})} \simeq u_f(x)$$

$$x \to \infty, \quad \frac{3\pi}{5} < \arg x < \frac{11\pi}{5}.$$

Quasilinear Stokes phenomenon

$$u_1(x) - u_0(x) = \frac{i}{\sqrt{\pi}} 2^{-11/8} 3^{-1/8} (-x)^{-1/8} e^{\frac{1}{5} 2^{11/4} 3^{1/4} (-x)^{5/4}}$$

$$\times \big(1 + O(x^{-3/8})\big),$$

$$x \to \infty, \quad \frac{3\pi}{5} \leq \arg x \leq \frac{7\pi}{5}$$

Asymptotics of the coefficients a_n.

$$a_n = -\frac{\sqrt{6}}{\sqrt{5}\pi^{3/2}}\left(\frac{1}{5}2^{11/4}3^{1/4}\right)^{-2n}\Gamma\left(2n-\frac{1}{2}\right)\left(1+O(n^{-3/10})\right)$$

$$+O(\rho^{5n}),$$

$$n \to \infty$$

Tronquée solutions,

$$u^{(1)}(x) \simeq u_f(x),$$

$$x \to \infty, \quad \frac{3\pi}{5} < \arg x < \frac{7\pi}{5}.$$

$$u^{(1)}(x) \equiv u^{(1)}(x; c_+) \equiv u^{(1)}(x; c_-)$$

$$u^{(1)}(x) - u_0(x) = c_+(-x)^{-1/8} e^{\frac{1}{5}2^{11/4}3^{1/4}(-x)^{5/4}} \times \left(1 + O(x^{-3/8})\right),$$

$$x \to \infty, \quad \frac{3\pi}{5} \le \arg x < \frac{7\pi}{5}$$

$$u^{(1)}(x) = \sqrt{-\frac{x}{6}} + O\left(x^{-2}\right) + c_+(-x)^{-1/8} e^{\frac{1}{5}2^{11/4}3^{1/4}(-x)^{5/4}} \times \left(1 + O(x^{-3/8})\right),$$

$$u^{(1)}(x) - u_1(x) = c_-(-x)^{-1/8} e^{\frac{1}{5} 2^{11/4} 3^{1/4} (-x)^{5/4}}$$

$$\times \left(1 + O(x^{-3/8})\right),$$

$$x \to \infty, \quad \frac{3\pi}{5} < \arg x \leq \frac{7\pi}{5}$$

$$u^{(1)}(x) = \sqrt{-\frac{x}{6}} + O\left(x^{-2}\right) + c_-(-x)^{-1/8} e^{\frac{1}{5} 2^{11/4} 3^{1/4} (-x)^{5/4}}$$

$$\times \left(1 + O(x^{-3/8})\right),$$

Quasiliniear Stokes phenomenon:

$$c_+ - c_- = \frac{i}{\sqrt{\pi}} 2^{-11/8} 3^{-1/8}$$

ANDREI'S LATER WORKS

The Riemann-Hilbert Method

1993 - 2016

- The complete lists of the asymptotics in the complex domain of the solutions of the first, second and fourth Painlevé equations supplemented by the complete lists of all the relevant connection formulae.
- The complete description of all possible scaling limits (double asymptotics in argument and parameter) of the first, the second and the forth Painlevé equations. All on the level of solutions.

- Monodromy approach to the scaling limits in isomonodrtomy systems
- Quasilinear Stokes phenomenon.
- The Riemann-Hilbert approach to the bi-orthogonal polynomials
- Isomonodromy deformation equations **without** Painlevé property
- Tracy-Widom distributions for general beta

SCALING LIMITS IN THE SECOND PAINLEVÉ EQUATION

1993, 2001

$$u_{xx} = xu + 2u^3 - \alpha,$$

$$x = \delta^{-2/3} t_0 + \delta^{1/3} t, \quad u = \delta^{-1/3} v, \quad \alpha = -4\delta^{-1} + \beta$$

$$\Downarrow$$

$$v_{tt} = 2v^3 + t_0 v + 4 + \delta(tv - \beta).$$

$$\delta \to 0$$

Formal result:

$$v_t^2 = \left(v^2 + \frac{t_0}{2}\right)^2 + 8V + a_0$$

where $a_0 \equiv a_0(t_0)$ is defined by the Boutroux equations

$$\Im\left\{\int_{\mathcal{L}} \sqrt{z^2 + \frac{t_0}{2}z^2 - \frac{a_0}{4}z + 1}\,\frac{dz}{z}\right\} = 0.$$

Everything depends on where is t_0 on the complex plane.

Special cases of the Boutroux equation:

- $t_0 = -6e^{\frac{2\pi i}{3}k}, \quad a_0 = -12e^{\frac{4i\pi}{3}k}$
- $t_0 = -4s - 2s^{-2}, \quad a_0 = -4s(s + 2s^{-2})$:

$$\Im\left\{\frac{t_0}{3}\sqrt{s - s^{-2}} - i\log\frac{\sqrt{s - s^{-2}} - is^{-1}}{\sqrt{s - s^{-2}} + is^{-1}}\right\} = 0$$

On the t_0 - complex plane this is the union of the rays

$$t_0 = -6e^{\frac{2\pi i}{3}k}\rho, \quad \rho \geq 1$$

and five branches asymptotic to

$$\Im\left(\frac{1}{3\sqrt{2}}t_0^{3/2} \pm \frac{3i}{2}\log t_0\right) = \text{const}$$

Consider the domains D_1, D_2, D_3. Then, we have,

- $t_0 \in D_j$: the asymptotics is elliptic
- $t_0 \in \partial D_j \setminus \left\{-6e^{\frac{2\pi i}{3}k}\right\}$: the asymptotics is trigonometric.
- near the vertices $t_0 = -6e^{\frac{2\pi i}{3}k}$: there exists an additional double scaling limit to PI:

$$x = -6\delta^{-2/3}+\delta^{2/15}\tau, \quad u = \delta^{-1/3}+\delta^{1/15}y, \quad \alpha = -4\delta^{-1}-\delta^{-1/5}\gamma$$

$$\Downarrow \quad \delta \to 0$$

$$y_{\tau\tau} = 6y^2 + \tau + \gamma$$

Andrei's principal achievement:

All the asymptotics are explicitly parametrized by the monodromy data $s = (s_1, s_2, s_3)$