Isomonodromy Deformations at a Resonant irregular Singularity

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Setting

- Linear $n \times n$ differential system:

$$\frac{dY}{dz} = \left(\Lambda(t) + \frac{A_1(t)}{z} \right) Y, \qquad z = \infty \text{ irregular}, \quad z = 0 \text{ Fuchsian},$$

depending *holomorphically* on parameter $t = (t_1, \dots, t_m)$ in a polydisc at 0:

$$\mathcal{U}_\epsilon(0) := \left\{ t \in \mathbb{C}^m \text{ such that } |t| := \max_{1 \leq i \leq m} |t_i| \leq \epsilon \right\}, \quad \epsilon > 0.$$

$$\Lambda(t) = \begin{pmatrix} u_1(t) & & & \\ & u_2(t) & & \\ & & \ddots & \\ & & & u_n(t) \end{pmatrix} \quad \text{diagonal}$$

This system has applications in:

- $n = 2$, classical special functions (confluent hypergeometric, Whittaker)
- $n = 3$; isomonodromic system associated with the sixth Painlevé equation [B. Dubrovin '96, M. Mazzocco 2002]
- Isomonodromic description of semisimple Frobenius manifolds [B. Dubrovin '96, '98]. *If pairwise distinct*, eigenvaues $u_1, ..., u_n$ are local (canonical) coordinates

$$\Lambda(t) = \mathrm{diag}(u_1(t), u_2(t), ..., u_n(t)).$$

Our problem: Coalescence of eigenvalues $u_a - u_b \to 0$ in the space of parameters.

Coalescence locus in the polydisc $\mathcal{U}_\epsilon(0)$:

$$\Delta := \{t \in \mathcal{U}_\epsilon(0) \text{ such that } u_a(t) = u_b(t) \text{ for some } 1 \leq a \neq b \leq n\}$$

We assune $t = 0 \in \Delta$ (no loss of generality).

Questions

Coalescence ($\longleftrightarrow$ "non-admissible" deformations!) corresponds to:

– fixed singularities of the Painlevé equation;
– points of the Frobenius manifold where the analytic theory (based on semisimplicity) fails.

Questions

1) If $A_1(t)$ is holomorphic in $\mathcal{U}_\epsilon(0)$, what can we say about the behaviour of fundamental solutions and monodromy data at Δ?

2) In isomonodromic case, can we prove that constant monodromy data are defined on the whole $\mathcal{U}_\epsilon(0)$? Can we compute them starting from the system at $t = 0 \in \Delta$?

3) In the isomonodromic case, if $A_1(t)$ is holomorphic in a small domain $\mathcal{B} \subset \mathcal{U}_\epsilon(0)$, with $\mathcal{B} \cap \Delta = \emptyset$, when can we say when Δ *is not* a branching locus for solutions?

Motivating questions:

• Frobenius manifolds. Locally constant monodromy data of the system locally parametrize a chart; they allow to re-construct of the whole manifold structure through an action of braid group and inverse RH problem.

For some important Frobenius manifolds (e.g. quantum cohomologies of almost all Grassmannians) the manifold structure is explicitly known only at coalescence points.

How can we compute monodromy data for the whole manifold staring from a coalescence point?

• Monodromy data of Painvevé transcendents remaining holomorphic at a fixed singularity (A_1 holomorphic in $\mathcal{U}_\epsilon(0)$) can be computed starting from the coalescence point $t = 0$?

• We need to extend the deformation theory at coelescence points, in a holomorphic way.

(Incomplete) list of bibliographic references for *resonant irregular singularities*, their unfolding and confluence of regular singularities.

R. Garnier (1919), J. P. Ramis (1989), A. Duval (1991),

A. A. Bolibruch (1994-8), D. G. Babbitt & V. S. Varadarajan (1985),

R. Schäfke (2001), A. A. Glutsyuk (1999-2004),

C. Lambert & C. Rousseau (2012),

J. Hurtubise & C. Lambert & C. Rousseau (2014), M. Klimes (2013-16),

M. Bertola & M. Y. Mo (2005),

M. V. Fedoryuk (1992), Y. P. Bibilo (2012),

P. P. Boalch (2012), T. Bridgeland & V. Toledano Laredo (2013),

etc...

!! The case of $\Lambda(t)$ *remaining diagonal* at coalescing eigenvalues (no change of Jordan type) seems to be missing from the existing literature.

Well known facts away from Δ, for $A_1(t)$ holomorphic [Sibuya/ Balser-Jurkat-Lutz]:

1) For $t \notin \Delta$, $\exists$ unique formal (holomorphic) matrix solution det $Y_F \neq 0$,

$$Y_F(z,t) = \left(I + \sum_{k=1}^{\infty} F_k(t) z^{-k} \right) z^{\mathrm{diag}(A_1(t))} e^{\Lambda(t) z}.$$

2) For $\mathcal{B} \subset \mathcal{U}_\epsilon(0)$ sufficiently small domain, s.t. $\mathcal{B} \cap \Delta = \emptyset$ there exist a (wide) sector $\mathcal{S}(\mathcal{B})$ and a unique fundamental matrix solution

$$Y(z,t) \sim Y_F(z,t), \quad z \to \infty, \quad z \in \mathcal{S}(\mathcal{B}), \quad \text{uniformly in } t \in \overline{\mathcal{B}}.$$

Our problem:

- If $\Delta \neq \emptyset$ there are problematic issues with the singular behaviour at Δ of fundamental matrix solutions and monodromy data.

- When t leaves $\mathcal{B}$ asymptotic expansions fail (in prescribed sectors).

Elementary example of problematic issues

$$\frac{dY}{dz} = \left[\begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix} + \frac{1}{z}\begin{pmatrix} 1 & 0 \\ t & 2 \end{pmatrix}\right] Y, \quad u_1 = 0, \; u_2 = t.$$

The coalescence locus is $\Delta = \{0\}$. Fundamental matrix solutions.
At fixed $t = 0$:

$$\mathring{Y}(z) = I \cdot \begin{bmatrix} z & 0 \\ 0 & z^2 \end{bmatrix}, \qquad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

For any $t \neq 0$:

$$Y(z,t) := \left[I + \begin{pmatrix} 0 & 0 \\ \mathcal{G}(z,t) & 0 \end{pmatrix}\right] \begin{pmatrix} z & 0 \\ 0 & z^2 e^{tz} \end{pmatrix},$$

$$\mathcal{G}(z,t) := t\Big[\, tz\; e^{tz}\mathrm{Ei}(tz) - 1\Big],$$

$$\mathrm{Ei}\,(z) := \int_z^{\infty} \frac{e^{-\zeta}}{\zeta} d\zeta = \int_1^{\infty} \frac{e^{-z\xi}}{\xi} d\xi \quad \text{exponential integral.}$$

Canonical asymptotic expansion

$$\mathcal{G}(z,t) \sim \sum_{n=1}^{\infty} \frac{(-1)^k k!}{t^{k-1}} \frac{1}{z^k}, \quad z \to \infty, \quad -\frac{3\pi}{2} < \arg(zt) < \frac{3\pi}{2},$$

Thus

$$Y(z,t) \sim \left(I + \sum_{k=1}^{\infty} F_k(t) z^{-k} \right) \begin{pmatrix} z & 0 \\ 0 & z^2 e^{tz} \end{pmatrix}$$

- Sector depends on t.
- For $t \in \mathcal{B}$ sufficiently small ($0 \notin \mathcal{B}$), the asymptotics holds in a t-independent sector $\mathcal{S}(\mathcal{B}) = S(\alpha, \beta) := \{\alpha < \arg z < \beta\}$, $\beta - \alpha > \pi$.

"Problematic" issues:

- $\Delta = \{t = 0\}$ is a *branching locus* for $Y(z,t)$, because $\mathrm{Ei}(tz)$ has a logarithmic branching at $t = 0$.
- $F_1(t)$ is holomorphic, but the other $F_k(t)$'s, $k \geq 2$, *have a pole at* $t = 0$ of order t^{-k+1}.
- Boundary rays of the sector where the asymptotics holds rotate with t moving around $\Delta = \{t = 0\}$ (they rotate).

Example, $n = 2$.

Isomonodromic case $\Longrightarrow u \equiv t$ [Jimbo-Miwa-Ueno '81].

$$\frac{dY}{dz} = \left[\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} + \frac{A_1(u)}{z} \right] Y, \qquad \Delta = \{u_1 = u_2\}.$$

Jimbo-Miwa-Ueno isomonodromy deformation equations $\longrightarrow$ determine A_1:

$$\frac{\partial A_1}{\partial u_1} = [[F_1, E_1], A_1], \qquad \frac{\partial A_1}{\partial u_2} = [[F_1, E_2], A_1],$$

$(E_k)_{ab} = \delta_{ka}\delta_{kb}$,
$(F_1)_{ab} = (A_1)_{ab}/(u_b - u_a)$, $a \neq b$, and $(F_1)_{aa} = -\sum_{b\neq a}(A_1)_{ab}(F_1)_{ba}$,
$a, b \in \{1, 2\}$.

$$\Downarrow$$

$$A_1(u) = \begin{pmatrix} a & c(u_1 - u_2)^{-b} \\ d(u_1 - u_2)^b & a - b \end{pmatrix}, \qquad a, b, c, d \in \mathbb{C}.$$

For generic values of the parameters, *A_1 is singular* $(b, c, d \neq 0)$ and *Δ is a branching locus* $(b \notin \mathbb{Z})$

Away from Δ (i.e. $u_1 \neq u_2$):

$$Y_F(z,u) = \left(I + \frac{F_1(u)}{z} + \frac{F_2(u)}{z^2} + \cdots \right) z^{\mathrm{diag}(A_1)} \begin{pmatrix} e^{u_1 z} & 0 \\ 0 & e^{u_2 z} \end{pmatrix},$$

$F_k(u)$ uniquely determined. (e.g. $(F_1)_{12} = -c(u_1 - u_2)^{-b-1}$)

System solved in terms of Whittaker functions: Look for vector solution $\begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix}$. Substitution and elimination of y_2: 2nd order ODE for y_1.

$$y_1(z) = e^{\frac{1}{2}(u_1+u_2)z} z^{a-\frac{b+1}{2}} w(x), \quad x = z(u_1 - u_2),$$

$$\frac{d^2 w}{dx^2} + \left(-\frac{1}{4} + \frac{\kappa}{x} + \frac{\frac{1}{4} - \mu^2}{x^2} \right) w = 0; \quad \mu^2 := \frac{b^2 + 4cd}{4}, \quad \kappa := -\frac{1+b}{2}.$$

We can explicitly write fundamental matrix solutions in terms of Whittaker functions.

Using the asymptotic properties of Whittaker functions ($x \to \infty$)

$$W_{\kappa,\mu}(x) = x^{\kappa} e^{-x/2}\left(1 + O\left(\frac{1}{x}\right)\right), \quad -\frac{3\pi}{2} < \arg x < \frac{3\pi}{2}, \quad x = (u_1 - u_2)z,$$

we show that we can pick up three (unique) fundamental solutions

$$Y_1(z,u), \quad Y_2(z,u), \quad Y_3(z,u),$$

such that

$$Y_\nu(z,u) \sim Y_F(z,u), \quad z \to \infty \text{ and } u_1 - u_2 \neq 0,$$

for $x = (u_1 - u_2)z$ in the overlapping sectors:

$$\mathcal{S}_1 = S\Big(-\frac{5\pi}{2}, -\frac{\pi}{2}\Big), \quad \mathcal{S}_2 = S\Big(-\frac{3\pi}{2}, \frac{\pi}{2}\Big), \quad \mathcal{S}_3 = S\Big(-\frac{\pi}{2}, \frac{3\pi}{2}\Big).$$

Fact 1: Sectors change with u.
Fact 2: $\Delta = \{u_1 - u_2 = 0\}$ is a branching locus for the $Y_\nu(z,u)$'s. Indeed, Whittaker functions depend on $x = z(u_1 - u_2)$, and are multivalued at $x = 0$.

Stokes matrices $\mathbb{S}_1$, $\mathbb{S}_2$*:* Using cyclic relations of Whittaker functions we compute

$$Y_3(z,u)=Y_2(z,u)\ \mathbb{S}_2, \quad Y_2(z,u)=Y_1(z,u)\ \mathbb{S}_1,$$

$$\mathbb{S}_2=\begin{pmatrix} 1 & 0 \\ s_2 & 1 \end{pmatrix}, \quad \mathbb{S}_1=\begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix},$$

$$s_2=\frac{2\pi i}{c\ \Gamma\left(\frac{\sqrt{b^2+4cd}}{2}-\frac{b}{2}\right)\ \Gamma\left(-\frac{\sqrt{b^2+4cd}}{2}-\frac{b}{2}\right)},$$

$$s_1=\frac{-2\pi i\ c\ e^{i\pi b}}{\Gamma\left(\frac{\sqrt{b^2+4cd}}{2}+1+\frac{b}{2}\right)\ \Gamma\left(-\frac{\sqrt{b^2+4cd}}{2}+1+\frac{b}{2}\right)}.$$

Lemma: *Stokes matrices are trivial* ($s_1=s_2=0$) $\Longleftrightarrow$ *one of the following holds:*

1) $c=d=0$ *and* $b\in\mathbb{C}$,
2) $cd=MN$, $\quad b=N-M$,
3) *either* $d=0$ *and* $b=-M$, *or* $c=0$ *and* $b=N$.

$M,N\geq 1$ *integers.*

Using Lemma and monodromy of Whittaker functions and $x = 0$:

entries of $Y_\nu(z,u) \cong z^a x^{\frac{1}{2}+\kappa+\mu}(C_+ + O(x)) + z^a x^{\frac{1}{2}+\kappa-\mu}(C_- + O(x)), \quad x = z(u_1 - u_2)$,

we can prove (by computations!) the following

Fact 3: *If $A_1(u)$ is holomorphic at Δ (choose b, c, d) and both*

$$(A_1)_{12} \quad \text{and} \quad (A_1)_{21} = \mathcal{O}(u_1 - u_2) \to 0 \text{ for } u_1 - u_2 \to 0,$$

then: $Y_\nu(z,u)$'s, $\nu = 1, 2, 3$, are holomorphic at Δ. Moreover, $s_1 = s_2 = 0$.

Conversely:

Fact 4: *If $s_1 = s_2 = 0$,*
then $A_1(u)$ and $Y_\nu(z,u)$, $\nu = 1, 2, 3$, are single valued for a loop $(u_1 - u_2) \mapsto (u_1 - u_2)e^{2\pi i}$ around the coalescence locus $u_1 = u_2$. Namely, Δ is not a branching locus.

$A_1(u)$ and $Y_\nu(z,u)$ may have poles at $u_1 - u_2 = 0$

Facts 1, 2, 3, 4 actually hold in the general case!

Our goal in the general case

(1) [Non-isomonodromic case] If A_1 holomorphic in $\mathcal{U}_\epsilon(0)$, find sufficient conditions such that:

• solutions can be holomorphically t-continued to the whole polydisc $\mathcal{U}_\epsilon(0)$ preserving the asymptotic relations $Y_\nu \sim Y_F$ in wide sectors, independent of t;

• $\exists \lim_{t\to t_\Delta} \mathbb{S}_\nu(t)$, $t_\Delta \in \Delta$ (e.g. does $\mathbb{S}_\nu(0)$ exist?).

(2) [Isomonodromic case – I] If A_1 holomorphic in $\mathcal{U}_\epsilon(0)$, extend the *isomonodromy deformation theory*, initially defined in a small domain $\mathcal{B}$, to the whole $\mathcal{U}_\epsilon(0)$, in order to

• prove when monodromy data are constant on the whole $\mathcal{U}_\epsilon(0)$;

• show that they can be computed *just by considering* the linear system at $0 \in \Delta$ (the simplest system!)

(3) [Isomonodromic case – II]

• if A_1 is holomorphic *only* in a small domain $\mathcal{B}$, find suitable conditions on Stokes matrices such that Δ is not a branching locus.

In this talk, we mainly explain results concerning points (2), (3). Details about (1) are in arXiv:1706.04808

Extension of $\mathcal{B}$ and Cells Decomposition

Recall again Sibuya's result:

$$Y(z,t) \sim Y_F(z,t), \quad z \to \infty, \quad z \in \mathcal{S}(\mathcal{B}), \quad \text{uniformly in } t \in \overline{\mathcal{B}}.$$

$$\mathcal{B} \subset \mathcal{U}_\epsilon(0) \text{ is (very) } \underline{\text{small}}, \text{ s.t. } \underline{\mathcal{B} \cap \Delta = \emptyset}.$$

We want to extend the deformation theory: from small $\mathcal{B}$ to the whole polydisc $\mathcal{U}_\epsilon(0)$, including Δ.

Some ingredients... In the universal covering of z-plane $\widetilde{\mathbb{C}\backslash\{0\}}$ we define:

- Stokes rays associated with $\Lambda(t)$:

$$z \in \widetilde{\mathbb{C}\backslash\{0\}} \text{ s.t. } \Re e(u_a(t) - u_b(t))z = 0 \quad (u_a(t) \neq u_b(t),\ 1 \leq a \neq b \leq n).$$

They rotate as t varies.

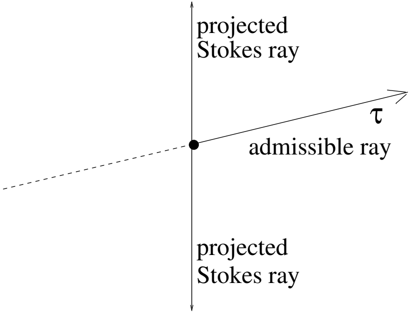
- Admissible ray $\arg z = \tau$: a ray of direction τ, that does not coincide with any Stokes ray *associated with* $\Lambda(0)$.

Example:

$\Lambda(t) = \text{diag}(0, t, 1),\ t$ small.

for $\Lambda(0)$:

Stokes rays obtained from $\Re e(tz) = 0$.
2 rays $(+2k\pi)$.

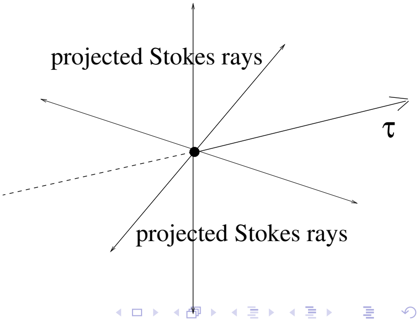


for $\Lambda(t)$:

Stokes rays obtained from

$\Re e(tz) = 0$, $\Re e((t-1)z) = 0$.

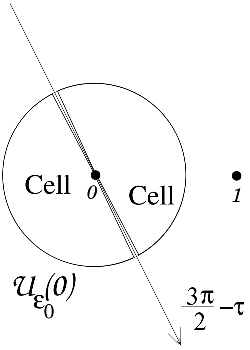
6 rays $(+2k\pi)$.



Stokes rays rotate with varying t.

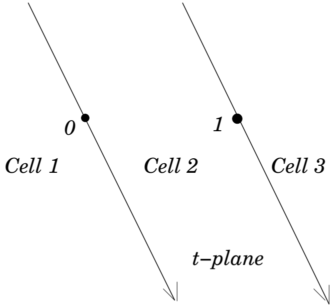
• Crossing locus $X(\tau)$ in $\mathcal{U}_\epsilon(0)$ is such that $t \in X(\tau)$ iff some Stokes rays associated with $\Lambda(t)$ coincide with / cross the admissible ray $\arg z = \tau$.
• τ-cell in $\mathcal{U}_\epsilon(0)$: any connected component of $\mathcal{U}_\epsilon(0) \backslash (\Delta \cup X(\tau))$.

Example:
$\Lambda(t) = \text{diag}(0, t, 1)$, t small



Locally, in $\mathcal{U}_\epsilon(0)$

In this example, also global cells

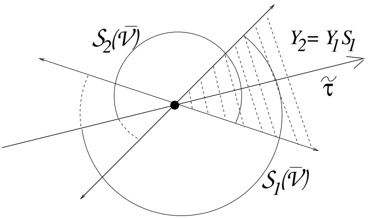


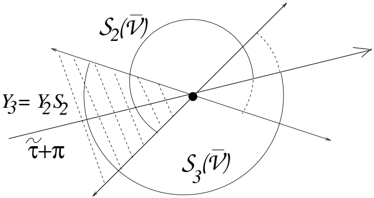
Proposition: If A_1 is holomorphic in $\mathcal{U}_\epsilon(0)$, *$\mathcal{B}$ can be any simply connected domain such that $\overline{\mathcal{B}}$ lies in a τ-cell.*
There are actual fundamental solutions $Y_\nu(z,t)$, $\nu \in \mathbb{Z}$, holomorphic in $\widetilde{\mathbb{C}\backslash\{0\}} \times \mathcal{B}$, uniquely determined by

$$Y_\nu(z,t) \sim Y_F(z,t), \quad z \to \infty, \quad t \in \mathcal{B},$$

uniformly on $\overline{\mathcal{B}}$, in suitable overlapping sectors $\mathcal{S}_\nu(\mathcal{B}) \subset \widetilde{\mathbb{C}\backslash\{0\}}$, with angular opening $> \pi$.

$$Y_{\nu+1}(z,t) = Y_\nu(z,t)\, \mathbb{S}_\nu(t) \quad \longleftarrow \quad \text{Stokes matrices holomorphic in } \mathcal{B}$$





Definition of essential essential monodromy data (in red)

If $t \in \mathcal{B}$, it make sense to define essential monoromy data.

1) At $z = 0$ (Fuchsian sing.): fundamental solution in Levelt form

$$Y^{(0)}(z,t) = G(t)\left(I + \sum_{\ell=1}^{\infty} \Psi_\ell(t) z^\ell\right) z^D z^{S+R}, \qquad t \in \mathcal{U}_\epsilon(0),$$

R nilpotent, $G^{-1}A_1 G = S + D$ Jordan form of A_1, with D diagonal matrix of integers, $\mathrm{diag}(S) = (s_1, \ldots, s_n)$, $0 \le \Re\mathfrak{e}\ s_j < 1$ (equivalently, the eigenvalues of A_1).

2) At $z = \infty$:

$$Y_F(z,t) = \left(I + \sum F_k z^{-k}(t)\right) z^{\mathrm{diag}(A_1)} e^{\Lambda(t) z}, \quad t \in \mathcal{B}$$

Take three fundamental matrices $Y_\nu(z,t)$, $\nu = 1, 2, 3$; $\quad Y_\nu(z,t) \sim Y_F(z,t)$ in $\mathcal{S}(\mathcal{B})$:

$$Y_2 = Y_1\ \mathbb{S}_1, \quad Y_3 = Y_2\ \mathbb{S}_2 \quad \longleftarrow \text{Stokes matrices } \mathbb{S}_1, \mathbb{S}_2.$$

$$Y_1 = Y^{(0)}\ C \qquad \longleftarrow \text{Connection matrix } C.$$

Isomonodromy deformations

Definition: The deformation is isomonodromic in $\mathcal{B}$ contained in a τ-cell (see Proposition) if the essential monodromy data can be taken independent of $t \in \mathcal{B}$.

In the isomonodromic case, $u_1, \ldots, u_n$ can be taken as the independent deformation parameters [Jimbo-Miwa-Ueno '81]. Thus we set

$$u_a(t) = u_a(0) + t_a, \qquad 1 \leq a \leq n = m$$

$$\Lambda(t) = \Lambda(0) + \mathrm{diag}(t_1, \ldots, t_n), \qquad t \in \mathcal{U}_\epsilon(0).$$

In this case:

Proposition: *τ-cells are topological cells, i.e. homeomorphic to balls in $\mathbb{R}^{2n}$. In particular, they are simply connected. Hence, $\mathcal{B}$ is any domain such that $\overline{\mathcal{B}}$ lies in a τ-cell.*

Isomonodromy deformations

Isomonodromy deformation equations

$$\frac{dY}{dz} = \left(\Lambda(t) + \frac{A_1(t)}{z}\right) Y, \qquad \left(\frac{\partial}{\partial t_k} = \frac{\partial}{\partial u_k}\right)$$

$$\frac{\partial Y}{\partial t_k} = \Omega_k(z,t) Y, \qquad k = 1, 2, ..., n$$

$$\Omega_k(z,t) := zE_k + W_k, \qquad W_k := \begin{pmatrix} 0 & 0 & -\frac{(A_1)_{1k}}{u_1 - u_k} & 0 & 0 \\ 0 & 0 & \vdots & 0 & 0 \\ \frac{(A_1)_{k1}}{u_k - u_1} & \cdots & 0 & \cdots & \frac{(A_1)_{kn}}{u_k - u_n} \\ 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & -\frac{(A_1)_{nk}}{u_n - u_k} & 0 & 0 \end{pmatrix}.$$

Compatibility

$$\frac{\partial A_1}{\partial t_k} = [W_k, A_1].$$

Problematic issues arise when
1) t crosses the boundary of a cell ("wall crossing");
2) t tends to a point of Δ.

- The expressions $\Re e\left[(u_a(t) - u_b(t))z\right]$, $1 \leq a \neq b \leq n$, have constant sign in $\mathcal{B}$, but vanish when a Stokes ray associated with $\Lambda(t)$ crosses the admissible direction $\tau \longleftrightarrow t$ crosses the boundary of a τ-cell.

$$\Downarrow$$

The asymptotics $Y_\nu(z,t) \sim Y_F(z,t)$ for $z \to \infty$ in $\mathcal{S}_\nu(\mathcal{B})$ may no longer hold for t outside the $\widetilde{\tau}$-cell containing $\mathcal{B}$.

- Δ is expected to be a branching locus for the $Y_\nu(z,t)$'s and $F_k(t)$'s.
- Monodromy data, in general, must be redefined at a point of Δ.

Main Results: non-isomonodromic case

Sufficient conditions in non-isomonodromic case (at $z = \infty$):

Theorem 0: *Let $A_1(t)$ be holomorphic in $\mathcal{U}_\epsilon(0)$, ϵ small (technical) and $\Lambda(t) = \Lambda(0) + \mathrm{diag}(t_1, ..., t_n)$. Assume that:*

- *all $F_k(t)$ are holomorphic at Δ (we have the explicit infinite sequence of necess. suff. conds. on A_1 for this);*
- *all $Y_\nu(z,u)$'s, $\nu \in \mathbb{Z}$ can be analytically continued on $\mathcal{U}_\epsilon(0)$.*

Then:

- *The asymptotic expansions*

$$Y_\nu(z,u) \sim Y_F(z,u)$$

hold for $t \in \mathcal{U}_\epsilon(0)$ in wider sectors $\widehat{\mathcal{S}}_\nu \supset \mathcal{S}_\nu(\mathcal{B})$;

- *Stokes matrices have the following properties*

$$\exists \lim_{t \to t_\Delta} \mathbb{S}_\nu(t) \text{ finite, } t \to t_\Delta \in \Delta;$$

$$(\mathbb{S}_\nu(t))_{ab} = (\mathbb{S}_\nu(t))_{ba} = 0 \text{ whenever } u_a(0) = u_b(0), \quad 1 \le a \neq b \le n.$$

Main Results: Isomonodromic Case

Theorem 1: *Let ϵ be small (technical) and A_1 holomorphic on $\mathcal{U}_\epsilon(0)$; let the deformation be isomonodromic in $\mathcal{B}$ contained in τ-cell. If*

$$(A_1(u(t)))_{ab} = \mathcal{O}(u_a - u_b) \to 0 \quad \text{as } t \text{ tends to a point of } \Delta$$

then:

- *The coefficients $F_k(t)$ of $Y_F(z,t)$ are holomorphic on $\mathcal{U}_\epsilon(0)$.*
- *The $Y_\nu(z,t)$'s, $\nu = 1,2,3$, can be t-analytically continued as single-valued holomorphic functions on $\mathcal{U}_\epsilon(0)$.*
- *The asymptotic expansions*

  $$Y_\nu(z,t) \sim Y_F(z,t) \quad \text{for } z \to \infty,$$

  hold for $t \in \mathcal{U}_\epsilon(0)$ in the wider sectors $\widehat{\mathcal{S}}_\nu \supset \mathcal{S}_\nu(\mathcal{B})$. The "wall crossing" does not affect the asymptotics.
- *The fundamental matrix solution $Y^{(0)}(z,t)$ is also t-analytically continued as a single-valued holomorphic function on $\mathcal{U}_\epsilon(0)$*

Main Results: extension of isomonod. deform. to $\mathcal{U}_\epsilon(0)$

- *The essential monodromy data R, $S + D$, $\mathbb{S}_1$, $\mathbb{S}_2$ and C are* globally defined and constant *on* $\mathcal{U}_\epsilon(0)$. *They coincide with the data of the system*

$$\frac{dY}{dz} = \left(\Lambda(0) + \frac{A_1(0)}{z}\right) Y.$$

- *Vanishing property:*

$$(\mathbb{S}_1)_{ab} = (\mathbb{S}_1)_{ba} = (\mathbb{S}_2)_{ab} = (\mathbb{S}_2)_{ba} = 0$$
$$\text{whenever } u_a(0) = u_b(0), \quad 1 \leq a \neq b \leq n.$$

□

Theorem is proved using isomonodromy deformation equations to show that there is a holomorphic t-continuation of the $Y_\nu(z,t)$'s. Then, we analyze how the asymptotic behaviour of the analytic continuation of any $Y_\nu(z,t)$ changes when Stokes rays cross $\arg z = \tau$: this gives information about Stokes matrices and wide sectors for the asymptotics.

Main Result

$$\frac{dY}{dz} = \left(\Lambda(0) + \frac{A_1(0)}{z}\right) Y \qquad (*)$$

has a family of formal solutions, since $\Lambda(0)$ has coalescing eigenv. This means that there is a family of Stokes matrices for given sectors.

Lemma *The formal solution of $(*)$ is unique $\iff$ diagonal entries of A_1 do not differ by non-zero integers. So Stokes matrices $\mathbb{S}_1, \mathbb{S}_2$ of $(*)$ are uniquely defined for given sectors.*

Corollary: *If diagonal entries of A_1 do not differ by non-zero integers, the monodromy data $\mathbb{S}_1, \mathbb{S}_2$, C, R, $D + S$ of*

$$\frac{dY}{dz} = \left(\Lambda(t) + \frac{A_1(t)}{z}\right) Y, \qquad (1)$$

defined on the whole $\mathcal{U}_\epsilon(0)$, are obtained just by computing the data of

$$\frac{dY}{dz} = \left(\Lambda(0) + \frac{A_1(0)}{z}\right) Y. \qquad (2)$$

! Important for computations: (2) is simpler than (1), because $(A_1(0))_{ab} = 0$ whenever $u_a(0) = u_b(0)$!!

Main Result

The converse theorem (proved using a Riemann-Hilbert problem)...

Theorem 2: *Let $A_1(t)$ be holomorphic "only" on an open simply connected domain $\mathcal{B}$ in a τ-cell, where the deformation is isomonodromic (ϵ small (technical)). If the vanishing conditions*

$$(\mathbb{S}_1)_{ab} = (\mathbb{S}_1)_{ba} = (\mathbb{S}_2)_{ab} = (\mathbb{S}_2)_{ba} = 0 \quad \text{whenever } u_a(0) = u_b(0), \; 1 \leq a \neq b \leq n,$$

hold, then

- *$Y_\nu(z,t)$ and $A_1(t)$ admit single-valued meromorphic continuation on $\mathcal{U}_\epsilon(0)\backslash\Delta$, as functions of t. Δ is not a branching locus.*
- *If $t \in \mathcal{U}_\epsilon(0)\backslash\Delta$ is not a pole of $Y_r(z,t)$, then*

$$Y_\nu(z,t) \sim Y_F(z,t) \quad \text{for } z \to \infty \text{ in } \widehat{\mathcal{S}}_\nu(t) \supset \mathcal{S}_\nu(\mathcal{B}), \qquad \nu = 1,2,3,$$

and Stokes matrices are well defined and constant

$$Y_{\nu+1}(z,t) = Y_\nu(z,t)\, \mathbb{S}_\nu, \qquad \nu = 1,2.$$

Applications of Theorem 1: Painlevé equations

Example: Painlevé 6 is the isomonodromy deformation eq. of a 3×3 system.
Theorem 1 applies to Painlevé transcendents holomorphic at a fixed singularity.

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left[\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right]\left(\frac{dy}{dt}\right)^2 - \left[\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right]\frac{dy}{dt} + \\ + \frac{1}{2}\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[(2\mu-1)^2 + \frac{t(t-1)}{(y-t)^2}\right], \quad \mu \in \mathbb{C}.$$

Isomonodromic system: $Y(z) \mapsto e^{u_1 z} Y(z)$, and $z \to (u_3 - u_1)z$

$$\frac{dY}{dz} = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{A_1(t)}{z}\right] Y, \qquad t = \frac{u_2 - u_1}{u_3 - u_1},$$

$$A_1(t) =: \begin{pmatrix} 0 & \Omega_2 & -\Omega_3 \\ -\Omega_2 & 0 & \Omega_1 \\ \Omega_3 & -\Omega_1 & 0 \end{pmatrix}.$$

$$\Omega_1 = i\frac{\sqrt{y-1}\sqrt{y-t}}{\sqrt{t}}\left[\frac{Q/2}{(y-1)(y-t)}+\mu\right], \qquad \Omega_2 = i\frac{\sqrt{y}\sqrt{y-t}}{\sqrt{1-t}}\left[\frac{Q/2}{y(y-t)}+\mu\right],$$

$$\Omega_3 = -\frac{\sqrt{y}\sqrt{y-1}}{\sqrt{t}\sqrt{1-t}}\left[\frac{Q/2}{y(y-1)}+\mu\right], \qquad Q := t(t-1)\frac{dy}{dt} - y(y-1).$$

The A_3-algebraic solution, $\mu = -\frac{1}{4}$, obtained in [Dubrovin-Mazzocco 2000]

$$y(s) = \frac{(1-s)^2(1+3s)(9s^2-5)^2}{(1+s)(243s^6+1539s^4-207s^2+25)}, \quad t(s) = \frac{(1-s)^3(1+3s)}{(1+s)^3(1-3s)}, \quad s \in \mathbb{C}.$$

A holomorphic branch by letting $s \to -\frac{1}{3}$ (Taylor expansion). We find:

$$\Omega_1(t) = i\sqrt{2}\left(\frac{1}{8} - \frac{1}{256}t - \frac{17}{16384}t^2 - \frac{257}{524288}t^3 + O(t^4)\right),$$
$$\Omega_2(t) = -\frac{1}{32}t - \frac{1}{64}t^2 - \frac{173}{16384}t^3 + O(t^4),$$
$$\Omega_3(t) = i\sqrt{2}\left(\frac{1}{8} + \frac{1}{256}t + \frac{47}{16384}t^2 + \frac{1217}{524288}t^3 + O(t^4)\right).$$

Since $\lim_{t\to 0}\Omega_2(t) = 0$, Theorem holds.

Theorem holds: in oder to comput monodromy when $|t|$ is small, just put $t = 0$.

$$\frac{dY}{dz} = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{A_1(0)}{z} \right] Y, \quad A_1(0) = \begin{pmatrix} 0 & 0 & -i\sqrt{2}/8 \\ 0 & 0 & i\sqrt{2}/8 \\ i\sqrt{2}/8 & -i\sqrt{2}/8 & 0 \end{pmatrix}$$

Reduction to Bessel equation. Using Hanckel functions with sectors

$$\mathcal{S}_1 = S\Big(-\frac{3\pi}{2}, \frac{\pi}{2}\Big), \qquad \mathcal{S}_2 = S\Big(-\frac{\pi}{2}, \frac{3\pi}{2}\Big), \qquad \mathcal{S}_3 = S\Big(\frac{\pi}{2}, \frac{5\pi}{2}\Big).$$

we obtain

$$\mathbb{S}_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbb{S}_2 = \mathbb{S}_1^{-T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}.$$

Other data are obtained by an action of the braid group. The result agrees with that of Dubrovin-Mazzocco (method is due to Jimbo '82), but is obtained <u>much faster</u>.

Applications: Frobenius manifolds

Frobenius structure on $QH^\bullet(G(k,n))$ (no details, no definitions...).

For almost all Grassmannians [Cotti '16], we explicitly know the linear system only at some coalescence points. We can apply our theorem.

The whole manifold structure can be reconstructed (in principle) from the monodromy data *computed at a coalescence point*.

- *Simplest example:* For $QH^\bullet(G(2,4))$. $n = 6$.

$$\Lambda(0) = 4\sqrt{2} \cdot \mathrm{diag}(0, 0, -i, i, -1, 1) \qquad \longleftarrow \quad \text{coalescence}$$

$A_1(0)$ is explicitly known and $(A_1(0))_{12} = (A_1(0))_{21} = 0$.

We prove that $\Lambda(t)$, $A_1(t)$ are holomorphic at $t = 0$ $\Longrightarrow$ Theorem holds.

Computations of $\mathbb{S}_1$, $\mathbb{S}_2$, C can be *explicitly* done. Indeed, the system at $t = 0$ reduces to a generalized hypergeometric equation.
Up to some admissible transformations $\mathbb{S}_1 \mapsto \mathbb{S}$ (including action of braid group) we obtain

$$\mathbb{S}^{-1} = \begin{pmatrix} 1 & 4 & 10 & 6 & 20 & 20 \\ 0 & 1 & 4 & 4 & 16 & 20 \\ 0 & 0 & 1 & 0 & 4 & 10 \\ 0 & 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We have also computed C explicitly.

Explicit computation of monodromy data allows to prove or refine conjectures [Dubrovin 1998, 2013] prescribing an explicit relation between the monodromy data of quantum cohomology of smooth projective varieties and suitable quantities associated with objects of exceptional collections in derived categories of coherent sheaves on these varieties.

General results on the conjecture to appear [Cotti - Dubrovin - G. on arXiv 1709... or 1710...].

Thank you!