Painlevé equations and orthogonal polynomials

Walter Van Assche

KU Leuven, Belgium

Kapaev workshop, Ann Arbor MI, 28 August 2017

Contents

Painlevé equations (discrete and continuous) appear at various places in the theory of orthogonal polynomials:

- Discrete Painlevé equations for the recurrence coefficients of orthogonal polynomials
- Painlevé differential equations for the recurrence coefficients (Toda flows)
- Rational solutions of Painlevé equations
- [Special function solutions of Painlevé equations]
- [Local asymptotics for orthogonal polynomials at critical points]

In preparation:
Orthogonal Polynomials and Painlevé Equations
Australian Mathematical Society Lecture Series

Orthogonal polynomials

$$\int p_n(x)p_m(x)w(x)\,dx = \delta_{m,n}, \qquad p_n(x) = \gamma_n x^n + \cdots$$

$$\sum_{k=0}^{\infty} p_n(x_k)p_m(x_k)w_k = \delta_{m,n}.$$

Three term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x), \qquad \text{orthonormal}$$

$$xP_n(x) = P_{n+1}(x) + b_nP_n(x) + a_n^2P_{n-1}(x), \qquad \text{monic polynomials}$$

Orthogonal polynomials on the unit circle (OPUC)

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi_n(z)\overline{\varphi_m(z)}v(\theta)\,d\theta = \delta_{m,n}, \qquad z = e^{i\theta}, \quad \varphi_n(z) = \kappa_n z^n + \cdots$$

$$z\Phi_n(z) = \Phi_{n+1}(z) + \overline{\alpha_n}\Phi_n^*(z), \qquad \Phi_n^*(z) = z^n\overline{\Phi}_n(1/z).$$

discrete Painlevé I

Let us consider $w(x) = e^{-x^4+tx^2}$ on $(-\infty, \infty)$.

The symmetry $w(-x) = w(x)$ implies $b_n = 0$, hence

$$xp_n(x) = a_{n+1}p_{n+1}(x) + a_n p_{n-1}(x).$$

The structure relation is

$$p_n'(x) = A_n p_{n-1}(x) + C_n p_{n-3}(x).$$

Compatibility between these two relations gives

$$4a_n^2 \left(a_{n+1}^2 + a_n^2 + a_{n-1}^2 - \frac{t}{2} \right) = n$$

This is known as **discrete Painlevé I** (d-$\mathrm{P_I}$).

Some history

- Already in old work of Laguerre (1885),
- Also in work of Shohat (1939),
- Independently found by Freud (1976),
- Positive solution analyzed by Nevai (1983).
- Asymptotic expansion by Máté-Nevai-Zaslavsky (1985)
- Recognized as $\mathrm{d}\text{-}\mathrm{P}_{\mathrm{I}}$ by Fokas-Its-Kitaev (1991).
- Relation with continuous Painlevé IV (Magnus, 1995)

Asymptotic behavior of a_n^2

Theorem (Freud)

The recurrence coefficients for the weight $w(x) = e^{-x^4+tx^2}$ satisfy

$$\lim_{n\to\infty} \frac{a_n}{n^{1/4}} = \frac{1}{\sqrt[4]{12}}.$$

Special solution

Put $x_n = a_n^2$, then (for $t = 0$)

$$x_n(x_{n+1} + x_n + x_{n-1}) = an, \qquad a = 1/4. \tag{1}$$

For orthogonal polynomials we want a solution with $x_0 = 0$ (because $a_0^2 = 0$) and $x_n > 0$ for $n \geq 1$.

Theorem (Lew and Quarles, Nevai)

There is a unique solution of (1) for which $x_0 = 0$ and $x_n > 0$ for all $n \geq 1$.

Langmuir lattice or Kac-van Moerbeke equations

General orthogonal polynomials

$$\int_{\mathbb{R}} P_n(x)P_m(x)\, d\mu(x) = 0, \qquad m \neq n.$$

Theorem

Let μ be a symmetric positive measure on $\mathbb{R}$ for which all the moments exist and let μ_t be the measure for which $d\mu_t(x) = e^{tx^2}\, d\mu(x)$, where $t \in \mathbb{R}$ is such that all the moments of μ_t exist. Then the recurrence coefficients of the orthogonal polynomials for μ_t satisfy the differential-difference equations

$$\frac{d}{dt} a_n^2 = a_n^2(a_{n+1}^2 - a_{n-1}^2), \qquad n \geq 1.$$

Painlevé IV

Put $a_n^2 = x_n$, then

$$\begin{aligned} n &= 4x_n(x_{n+1} + x_n + x_{n-1} - t/2), && (2) \\ x_n' &= x_n(x_{n+1} - x_{n-1}) && (3) \end{aligned}$$

Eliminate x_{n+1} and x_{n-1} using (2)–(3)

$$x_n'' = \frac{(x_n')^2}{2x_n} + \frac{3x_n^3}{2} - tx_n^2 + x_n\left(\frac{n}{2} + \frac{t^2}{8}\right) - \frac{n^2}{32x_n}$$

This is **Painlevé IV**.

Orthogonal polynomials on the imaginary line

The weight $e^{-x^4+tx^2}$ behaves well on $\mathbb{R}$, but it also tends to 0 along the imaginary axis $i\mathbb{R}$.

Orthogonal polynomials $(Q_n)_n$ on the imaginary axis

$$\int_{-i\infty}^{+i\infty} Q_n(x)Q_m(x)e^{-x^4+tx^2}\,dx = 0, \qquad n \neq m,$$

$$xQ_n(x) = Q_{n+1}(x) - b_nQ_{n-1}(x).$$

One has $Q_n(x;t) = (-i)^nP_n(ix;-t)$. Hence $b_n(t) = -a_n^2(-t)$.

The $(b_n)_n$ is the unique **negative** solution of $\mathrm{d\text{-}P_I}$ with $b_0 = 0$.

Orthogonal polynomials on the cross

We can combine $\mathbb{R}$ and $i\mathbb{R}$ and look for orthogonal polynomials $(R_n)_n$ for which $(n \neq m)$

$$\alpha \int_{-\infty}^{\infty} R_n(x)R_m(x)e^{-x^4+tx^2}\,dx + \beta \int_{-i\infty}^{+i\infty} R_n(x)R_m(x)e^{-x^4+tx^2}\,|dx| = 0,$$

with $\alpha, \beta > 0$.

$$xR_n(x) = R_{n+1}(x) - c_n R_{n-1}(x)$$

The $(c_n)_n$ still satisfy $\mathrm{d}\text{-}\mathrm{P_I}$, but with initial conditions

$$c_0 = 0, \quad c_1 = \frac{\alpha m_2(t) - \beta m_2(-t)}{\alpha m_0(t) + \beta m_0(-t)},$$

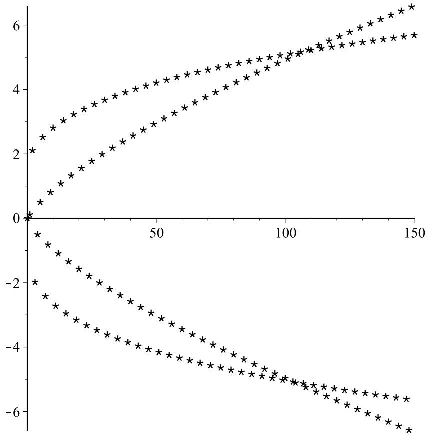
where

$$m_{2k}(t) = \int_{-\infty}^{\infty} x^{2k} e^{-x^4+tx^2}\,dx$$

are in terms of parabolic cylinder functions.

Orthogonal polynomials on the cross

The solution of $\mathrm{d}\text{-}\mathrm{P}_\mathrm{I}$ behaves very different. $(t = 0, \beta/\alpha = 1/2)$



Orthogonal polynomials on the cross

$$4x_n\left(x_{n+1}+x_n+x_{n-1}-\frac{t}{2}\right)=n$$

The case $\alpha=\beta$ and $t=0$ is special: the initial values are $x_0=x_1=0$.
0 is a singularity of $\mathrm{d}\text{-}\mathrm{P}_{\mathrm{I}}$ and gives $x_2=\infty$, hence R_3 does not exist. The singularity is confined to a finite number of terms.

Property

For $t=0$ and $\alpha=\beta$ one has that R_{4n-1} does not exist for $n\geq 1$. Furthermore

$$R_{4n}(x)=r_n(x^4),\quad R_{4n+1}(x)=xs_n(x^4),\qquad R_{4n+2}(x)=x^2s_n(x^4).$$

discrete Painlevé II

Let $v(\theta) = e^{t\cos\theta}$ on the unit circle

Trigonometric moments: modified Bessel functions

$$\frac{1}{2\pi}\int_0^{2\pi} e^{in\theta} v(\theta)\, d\theta = I_n(t),$$

$v(-\theta) = v(\theta)$ implies that $\alpha_n(t)$ are real valued.

$$v(\theta) = \hat{v}(z), \quad z = e^{i\theta}, \qquad \hat{v}(z) = \exp\left(t\frac{z+\frac{1}{z}}{2}\right).$$

Pearson equation

$$\hat{v}'(z) = \frac{t}{2}\left(1 - \frac{1}{z^2}\right)\hat{v}(z).$$

Structure relation

Property

The monic orthogonal polynomials for $v(\theta) = e^{t\cos\theta}$ *satisfy*

$$\Phi_n'(z) = n\Phi_{n-1}(z) + B_n\Phi_{n-2}(z),$$

for a sequence $(B_n)_n$*. In fact, one has*

$$B_n = \frac{t}{2}\frac{\kappa_{n-2}^2}{\kappa_n^2}.$$

Compatibility

$$z\Phi_n(z) = \Phi_{n+1}(z) + \alpha_n \Phi_n^*(x)$$

$$\Phi_n'(z) = n\Phi_{n-1}(z) + B_n \Phi_{n-2}(z)$$

Theorem (Periwal and Shevitz)

The Verblunsky coefficients for the weight $v(\theta) = e^{t\cos\theta}$ satisfy

$$-\frac{t}{2}(\alpha_{n+1} + \alpha_{n-1}) = \frac{(n+1)\alpha_n}{1-\alpha_n^2},$$

with initial values

$$\alpha_{-1} = -1, \quad \alpha_0 = \frac{I_1(t)}{I_0(t)}.$$

This is **discrete Painlevé II** (d-$\mathrm{P_{II}}$)

Discrete Painlevé II

Let $x_n = \alpha_{n-1}$, then

$$x_{n+1} + x_{n-1} = \frac{\alpha n x_n}{1 - x_n^2} \tag{4}$$

We need a solution with $x_0 = 1$ and $|x_n| < 1$ for $n \geq 1$.
Such a solution is unique.

Theorem

Suppose $\alpha > 0$. Then there is a unique solution of (4) for which $x_0 = 1$ and $-1 < x_n < 1$. The solution corresponds to $x_1 = I_1(2\alpha)/I_0(2/\alpha)$ and is positive for every $n \geq 0$.

Asymptotic behavior

This solution converges to zero (fast) $x_n \to 0$.

Property

The solution of $d\text{-}P_{II}$ *with* $x_0 = 1$ *and* $0 < x_n < 1$ *for* $n \geq 1$ *satisfies*

$$\frac{1}{\alpha^n n! \sum_{k=0}^n \frac{\alpha^{-k}}{k!}} \leq x_n \leq \frac{4^n n!}{\alpha^n (2n)!} \sim \frac{1}{\sqrt{2}} \left(\frac{e}{\alpha n}\right)^n.$$

The Ablowitz-Ladik lattice

The lattice equations corresponding to orthogonal polynomials on the unit circle are Ablowitz-Ladik lattice equations (or Schur flow).

Theorem

Let ν be a positive measure on the unit circle which is symmetric (the Verblunsky coefficients are real). Let ν_t be the modified measure $d\nu_t(\theta) = e^{t\cos\theta}\, d\nu(\theta)$, with $t \in \mathbb{R}$. The Verblunsky coefficients $(\alpha_n(t))_n$ for the measure ν_t then satisfy

$$2\alpha_n' = (1-\alpha_n^2)(\alpha_{n+1} - \alpha_{n-1}), \qquad n \geq 0.$$

Painlevé V and III

$\mathrm{d}\text{-}\mathrm{P}_{\mathrm{II}}$ gives

$$\alpha_{n+1} + \alpha_{n-1} = \frac{-2n\alpha_n}{t(1-\alpha_n^2)}$$

and Ablowitz-Ladik gives

$$\alpha_{n+1} - \alpha_{n-1} = \frac{2\alpha_n'}{1-\alpha_n^2}.$$

Eliminate α_{n+1} and α_{n-1} to find

$$\alpha_n'' = -\frac{\alpha_n}{1-\alpha_n^2}(\alpha_n')^2 - \frac{\alpha_n'}{t} - \alpha_n(1-\alpha_n^2) + \frac{(n+1)^2}{t^2}\frac{\alpha_n}{1-\alpha_n^2}.$$

If we put $\alpha_n = \frac{1+y}{1-y}$, then y satisfies **Painlevé V** with $\gamma = 0$.
The ratio $w_n = \alpha_n/\alpha_{n-1}$ satisfies **Painlevé III** [Hisakado, Tracy-Widom]

Generalized Charlier polynomials

Discrete orthogonal polynomials

$$\sum_{k=0}^{\infty} P_n(k)P_m(k)\frac{c^k}{(\beta)_k k!} = 0, \qquad n \neq m.$$

$$\begin{aligned} b_n + b_{n-1} - n + \beta &= \frac{cn}{a_n^2}, \\ (a_{n+1}^2 - c)(a_n^2 - c) &= c(b_n - n)(b_n - n + \beta - 1). \end{aligned}$$

This corresponds to a limiting case of **discrete Painlevé with surface/symmetry** $D_4^{(1)}$ in Sakai's classification.

Toda Lattice

Put $c = c_0 e^t$, then

$$\frac{c^k}{(\beta)_k k!} = e^{tk} \frac{c_0^k}{(\beta)_k k!}$$

Theorem

Suppose μ is a positive measure on the real line and $d\mu_t = e^{tx}\, d\mu(x)$, where t is such that all the moments of μ_t exist. Then the recurrence coefficients $b_n(t)$ and $a_n^2(t)$ of the orthogonal polynomials for the measure μ_t satisfy

$$\begin{aligned} \frac{d}{dt} a_n^2 &= a_n^2 (b_n - b_{n-1}), \qquad n \geq 1, \\ \frac{d}{dt} b_n &= a_{n+1}^2 - a_n^2, \qquad n \geq 0. \end{aligned}$$

generalized Charlier polynomials

Put $x_n = a_n^2$ and $y_n = b_n$ and $x_n' = dx_n/da$, $y_n' = dy_n/da$, then

$$\begin{aligned} (x_n - a)(x_{n+1} - a) &= a(y_n - n)(y_n - n + \beta - 1), \\ y_n + y_{n-1} - n + \beta &= \frac{an}{x_n} \end{aligned}$$

and the Toda lattice equations are

$$\begin{aligned} ax_n' &= x_n(y_n - y_{n-1}), \\ ay_n' &= x_{n+1} - x_n. \end{aligned}$$

Eliminate y_{n-1} and x_{n+1}, and put $x_n = \frac{a}{1-y}$, then $y(a)$ satisfies

$$y'' = \frac{1}{2}\left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{a} + \frac{(1-y)^2}{a^2}\left(\frac{n^2 y}{2} - \frac{(\beta-1)^2}{2y}\right) - \frac{2y}{a}$$

This is **Painlevé V** with $\delta = 0$.
This can be transformed to **Painlevé III**.

generalized Meixner polynomials

Discrete orthogonal polynomials

$$\sum_{k=0}^{\infty} P_n(k)P_m(k)\frac{(\gamma)_k a^k}{(\beta)_k k!} = 0, \qquad n \neq m.$$

Put $a_n^2 = na - (\gamma-1)u_n$, and $b_n = n + \gamma - \beta + a - \frac{\gamma-1}{a}v_n$, then

$$\begin{aligned}
(u_n + v_n)(u_{n+1} + v_n) &= \frac{\gamma-1}{a^2} v_n(v_n - a)\left(v_n - a\frac{\gamma-\beta}{\gamma-1}\right), \\
(u_n + v_n)(u_n + v_{n-1}) &= \frac{u_n}{u_n - \frac{an}{\gamma-1}}(u_n + a)\left(u_n + a\frac{\gamma-\beta}{\gamma-1}\right).
\end{aligned}$$

Initial values

$$a_0 = 0, \quad b_0 = \frac{\gamma a}{\beta}\frac{M(\gamma+1,\beta+1,a)}{M(\gamma,\beta,a)}$$

This is **asymmetric discrete Painlevé IV** or $\mathrm{d\text{-}P}(E_6^{(1)}/A_2^{(1)})$.

generalized Meixner

If we put

$$v_n(a) = \frac{a\Big(ay' - (1+\beta-2\gamma)y^2 + (n+1-a+\beta-2\gamma)y - n\Big)}{2(\gamma-1)(y-1)y},$$

then

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{a} + \frac{(y-1)^2}{a^2}\left(Ay + \frac{B}{y}\right) + \frac{Cy}{a} + \frac{Dy(y+1)}{y-1}$$

with

$$A = \frac{(\beta-1)^2}{2}, \quad B = -\frac{n^2}{2}, \quad C = n - \beta + 2\gamma, \quad D = -\frac{1}{2},$$

which is **Painlevé V**.

Other examples

Chen and Its (2010): $w(x) = x^{\alpha} e^{-x} e^{-t/x}$ on $[0, \infty)$
Put $b_n = 2n + \alpha + 1 + c_n$, $a_n^2 = n(n+\alpha) + y_n + \sum_{j=0}^{n-1} c_j$, and $c_n = 1/x_n$

$$\begin{aligned} x_n + x_{n-1} &= \frac{nt - (2n+\alpha)y_n}{y_n(y_n - t)} \\ y_n + y_{n+1} &= t - \frac{2n+\alpha+1}{x_n} - \frac{1}{x_n^2} \end{aligned}$$

$\mathrm{d\text{-}P}((2A_1)^{(1)}/D_6^{(1)})$

$$c_n'' = \frac{(c_n')^2}{c_n} - \frac{c_n'}{t} + (2n+\alpha+1)\frac{c_n^2}{t^2} + \frac{c_n^3}{t^2} + \frac{\alpha}{t} - \frac{1}{c_n}$$

which is **Painlevé III**.

Other examples

Basor, Chen, Ehrhardt (2010): $w(x) = (1-x)^\alpha (1+x)^\beta e^{-tx}$

$$\begin{aligned} tb_n &= 2n+1+\alpha+\beta-t-2R_n, \\ t(t+R_n)a_n^2 &= n(n+\beta)-(2n+\alpha+\beta)r_n - \frac{tr_n(r_n+\alpha)}{R_n} \end{aligned}$$

$$\begin{aligned} 2t(r_{n+1}+r_n) &= 4R_n^2 - 2R_n(2n+1+\alpha+\beta-t) - 2\alpha t, \\ n(n+\beta)-(2n+\alpha+\beta)r_n &= r_n(r_n+\alpha)\left(\frac{t^2}{R_nR_{n-1}} + \frac{t}{R_n} + \frac{t}{R_{n-1}}\right) \end{aligned}$$

and for $y = 1 + t/R_n$

$$y'' = \frac{3y-1}{2y(y-1)}(y')^2 - \frac{y'}{t} + 2(2n+1+\alpha+\beta)\frac{y}{t} - \frac{2y(y+1)}{y-1} + \frac{(y-1)^2}{t^2}\left(\frac{\alpha^2 y}{2} - \frac{\beta^2}{2y}\right)$$

which is **Painlevé V**.

Other examples

q-orthogonal polynomials:

$$w(x) = \frac{x^\alpha}{(-x^2;q^2)_\infty(-q^2/x^2;q^2)_\infty}, \qquad x \in [0,\infty)$$

gives rise to **q-discrete Painlevé III**

$$x_{n-1}x_{n+1} = \frac{(x_n + q^{-\alpha})^2}{(q^{n+\alpha}x_n + 1)^2}.$$

$$w(x) = \frac{x^\alpha(-p/x^2;q^2)_\infty}{(-x^2;q^2)_\infty(-q^2/x^2;q^2)_\infty}, \qquad x \in [0,\infty)$$

gives rise to **q-discrete Painlevé V**

$$(z_n z_{n-1} - 1)(z_n z_{n+1} - 1) = \frac{(z_n + \sqrt{q^{2-\alpha}/p})^2 (z_n\sqrt{pq^{\alpha-2}})^2}{(q^{n+\alpha/2-1}\sqrt{p}z_n + 1)^2}.$$

$$w(x) = x^\alpha(q^2x^2;q^2)_\infty, \qquad x \in \{q^k, k = 0,1,2,3,\ldots,\}$$

gives again **q-discrete Painlevé V**

Other examples

Bi-orthogonal polynomials on the unit circle (Forrester and Witte, 2006)

$$w(z) = z^{-\mu-\omega}(1+z)^{2\omega_1}(1+tz)^{2\mu} \begin{cases} 1, & \theta \notin (\pi-\phi,\pi), \\ 1-\xi, & \theta \in (\pi-\phi,\pi). \end{cases}, \quad t = e^{i\phi},$$

gives rise to

$$\begin{aligned} g_{n+1}g_n &= t\frac{(f_n+n)(f_n+n+2\mu)}{f_n(f_n-2\omega_1)}, \\ f_n + f_{n-1} &= 2\omega_1 + \frac{n-1+\mu+\omega}{g_n-1} + \frac{(n+\mu+\bar{\omega})t}{g_n-t} \end{aligned}$$

discrete Painlevé V (Sakai's surface $D_4^{(1)}$).

Other examples

Biane (2014) worked out a q-version

$$w(e^{i\theta}) = \left| \frac{(ae^{i\theta}; q)_\infty}{(be^{i\theta}; q)_\infty} \right|^2,$$

and found **discrete Painlevé equations corresponding to $A_3^{(1)}$**

$$r_{n+2}(a-bq^{n+2})+qr_n(a-bq^n) = \frac{2r_{n+1}}{1-r_{n+1}^2}\Big((1-q)(a+bq^{n+1})\beta_{n+1}+(1-q^{n+1})(ab+q)\Big),$$

with

$$\beta_n = \sum_{k=1}^{n} r_r r_{k-1}, \qquad r_n = \Phi_n(0).$$

Other examples

Witte (2015) worked out the most general case (deformed Askey-Wilson polynomials):

Pearson-type equation for the weight

$$\mathbb{D}_x w(x) = \frac{2V(x)}{W(x)} \mathbb{M}_x w(x),$$

where $\mathbb{D}_x$ is a divided difference operator on a special nonuniform lattice obtained from a quadratic equation

$$Ay^2 + 2Bxy + Cx^2 + 2Dy + 2Ex + F = 0,$$

and $\mathbb{M}_x$ is an average operator for the forward and backward moves on the lattice, and V, W are polynomials.

$$W(z) \pm \Delta y V(z) = z^{\mp 3} \prod_{j=1}^{6} (1 - a_j q^{-1/2} z^{\pm 1}),$$

gives the **q-discrete Painlevé cases for $E_7^{(1)}$** in Sakai's scheme.

Rational solutions of Painlevé equations

Rational solutions of Painlevé II

$$y'' = 2y^3 + xy + \alpha$$

has rational solutions if and only if $\alpha = n \in \mathbb{Z}$.
The solutions are

$$y_n = \frac{d}{dz} \log \frac{Q_{n-1}(z)}{Q_n(z)}$$

where Q_n are the Yablonskii-Vorobiev polynomials

$$Q_n(z) = c_n \det \begin{pmatrix} p_1 & p_3 & p_5 & \cdots & p_{2n-1} \\ p_1' & p_3' & p_5' & \cdots & p_{2n-1}' \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ p_1^{(n-1)} & p_3^{(n-1)} & p_5^{(n-1)} & \cdots & p_{2n-1}^{(n-1)} \end{pmatrix}$$

with polynomial p_k given by

$$\sum_{k=0}^{\infty} p_k(z)\lambda^k = \exp(z\lambda - \frac{4}{3}\lambda^3).$$

Rational solutions of Painlevé II

The polynomials p_k are multiple orthogonal polynomials on the 3-star

$$\Gamma = \bigcup_{k=0}^{2} \{z \in \mathbb{C} : \arg(z) = \frac{2\pi k}{3}\},$$

$$\int_\Gamma p_n(z) \mathrm{Ai}(2^{-2/3}|z|) z^k \, dz = 0, \qquad 0 \leq k \leq \lceil \frac{n}{2} \rceil - 1,$$

$$\int_\Gamma p_n(z) \frac{|z|}{z} \mathrm{Ai}'(2^{-2/3}|z|) z^k \, dz = 0, \qquad 0 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1.$$

Rational solutions of Painlevé II

Bertola and Bothner (2015) gave another determinant representation

$$Q_{n-1}^2(z) = d_n \det \begin{pmatrix} \mu_0(z) & \mu_1(z) & \mu_2(z) & \cdots & \mu_{n-1}(z) \\ \mu_1(z) & \mu_2(z) & \mu_3(z) & \cdots & \mu_n(z) \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \mu_{n-1}(z) & \mu_n(z) & \mu_{n+1}(z) & \ldots & \mu_{2n-2}(z) \end{pmatrix}$$

with $\mu_n(z) = p_n(2^{2/3}z)2^{-2n/3}$.

Advantage: this is a Hankel matrix, and hence allows to use the moment problem and orthogonal polynomials

$$P_n(x) = \frac{d_n}{Q_{n-1}^2(z)} \det \begin{pmatrix} \mu_0(z) & \mu_1(z) & \mu_2(z) & \cdots & \mu_n(z) \\ \mu_1(z) & \mu_2(z) & \mu_3(z) & \cdots & \mu_{n+1}(z) \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \mu_{n-1}(z) & \mu_n(z) & \mu_{n+1}(z) & \ldots & \mu_{2n-1}(z) \\ 1 & x & x^2 & \cdots & x^n \end{pmatrix}$$

Rational solutions of Painlevé II

z is a zero of Q_{n-1} (and a pole of the rational solution) if and only if the monic orthogonal polynomial $P_n(x)$ does not exist.

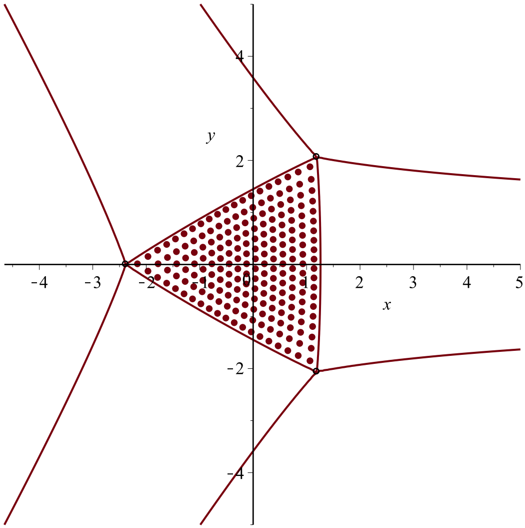
This allows to locate the zeros of Q_{n-1} by investigating the Riemann-Hilbert problem for the orthogonal polynomials $(P_k)_{k\in\mathbb{N}}$.

Theorem (Bertola-Bothner, Buckingham-Miller)

The zeros of $Q_n(n^{2/3}z)$ accumulate on a triangular shaped region with corners $x_k = -3/\sqrt[3]{2}e^{\frac{2\pi i}{3}k}$ $(k=0,1,2)$. The sides of the region are given by

$$\Re\left(-2\log\frac{1+\sqrt{1+2a^3}}{ia\sqrt{2a}} + \sqrt{1+2a^3}\frac{4a^3-1}{3a^3}\right) = 0,$$

where $a=a(x)$ is a solution of the cubic equation $1+2xa^2-4a^3=0$.



y
4
2
-2
-4
-4
-3
-2
-1
0
1
2
3
4
5
x

Rational solutions of Painlevé III

$$y'' = \frac{(y')^2}{y} - \frac{y'}{z} + \frac{\alpha y^2 + \beta}{z} + y^3 - \frac{1}{y}$$

Has rational solutions if and only if $\alpha + \beta = 4n$ or $\alpha - \beta = 4n$ for $n \in \mathbb{Z}$.

For $\alpha = 2n + 2\mu - 1$ and $\beta = 2n - 2\mu - 1$

$$y_n = 1 + \frac{d}{dz} \frac{S_{n-1}(z;\mu-1)}{S_n(z;\mu)}.$$

and for $\alpha = -2n + 2\mu - 1$ and $\beta = -2n - 2\mu + 1$

$$y_n = 1 - \frac{d}{dz} \frac{S_{n-1}(z;\mu)}{S_n(z;\mu-1)}.$$

Rational solutions of Painlevé III

$S_n(z;\mu)$ are Umemura polynomials

$$S_n(z;\mu) = c_n \det \begin{pmatrix} L_n^{\mu-n}(-z) & L_{n+1}^{\mu-n-1}(-z) & \cdots & L_{2n-1}^{\mu-2n+1}(-z) \\ L_{n-2}^{\mu-n+2}(-z) & L_{n-1}^{\mu-n+1}(-z) & \cdots & L_{2n-3}^{\mu-2n+3}(-z) \\ \vdots & \vdots & \cdots & \vdots \\ L_{-n+2}^{\mu+n-2}(-z) & L_{-n+3}^{\mu+n-3}(-z) & \cdots & L_1^{\mu-1}(-z) \end{pmatrix}$$

with $L_n^\alpha(x)$ the Laguerre polynomials.

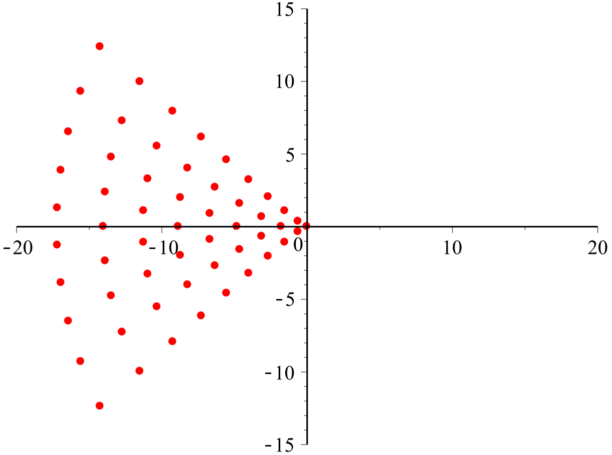
Rational solutions of Painlevé III

Wronskian formula:

$$S_n(z;\mu) = c_n \det \begin{pmatrix} L_1^{\mu-1}(-z) & L_3^{\mu-3}(-z) & \cdots & L_{2n-1}^{\mu-2n+1}(-z) \\ [L_1^{\mu-1}(-z)]' & [L_3^{\mu-3}(-z)]' & \cdots & [L_{2n-1}^{\mu-2n+1}(-z)]' \\ \vdots & \vdots & \cdots & \vdots \\ [L_1^{\mu-1}(-z)]^{(n-1)} & [L_3^{\mu-3}(-z)]^{(n-1)} & \cdots & [L_{2n-1}^{\mu-2n+1}(-z)]^{(n-1)} \end{pmatrix}$$

with $L_n^{\alpha}(x)$ the Laguerre polynomials.

Zeros of Umemura polynomial $S_{10}(\mu)$, $9 \geq \mu \geq -9$



Rational solutions of Painlevé IV

$$y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + 4zy^2 + 2(z^2-\alpha)y + \frac{\beta}{y}$$

Has rational solutions if and only if

$$\alpha = m, \quad \beta = -2(2n-m+1)^2$$

$$\alpha = m, \quad \beta = -2(2n-m+\frac{1}{3})^2.$$

$$\frac{d}{dz}\log\frac{H_{m+1,n}(z)}{H_{m,n}}, \quad \frac{d}{dz}\log\frac{H_{m,n}(z)}{H_{m,n+1}(z)}, \quad -2z+\frac{d}{d}\log\frac{H_{m,n+1}(z)}{H_{m+1,n}(z)}$$

with $H_{m,n}$ the generalized Hermite polynomials

$$-\frac{2}{3}z+\frac{d}{dz}\log\frac{Q_{m+1,n}(z)}{Q_{m,n}(z)}, \quad -\frac{2}{3}z+\frac{d}{dz}\log\frac{Q_{m,n}(z)}{Q_{m,n+1}(z)}, \quad -\frac{2}{3}z+\frac{d}{dz}\log\frac{Q_{m,n+1}(z)}{Q_{m+1,n}(z)}$$

with $Q_{m,n}$ the generalized Okamoto polynomials

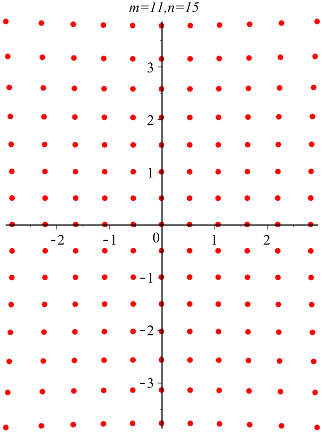
Rational solutions of Painlevé IV

The generalized Hermite polynomials are

$$H_{m,n}(z) = \det \begin{pmatrix} H_m(z) & H_{m+1}(z) & \cdots & H_{m+n-1}(z) \\ H_{m+1}(z) & H_{m+2}(z) & \cdots & H_{m+n}(z) \\ \vdots & \vdots & \cdots & \vdots \\ H_{m+n-1}(z) & H_{m+n}(z) & \cdots & H_{m+2n-2}(z) \end{pmatrix}$$

and $H_m(z)$ are Hermite polynomials.

Zeros of generalized Hermite polynomials



Zeros of generalized Hermite polynomials

Theorem (Buckingham)

The zeros of the generalized Hermite polynomial $H_{m,n}(m^{1/2}z)$ for the rational solution of Painlevé IV accumulate in a square shaped region with corners given by four solutions of the equation

$$r^4x^8-24r^2(r^3+r+1)x^4+32r(2r^3+3r^2-3r-2)x^2-48(r^2+r+1)^2=0$$

(not real or imaginary), and sides given by the curves

$$\Re\left(\frac{(1+r)r^{1/2}x}{2}R-(1+r)\log\left(2R-\frac{4}{(1+r)Q}-S\right)\right.$$
$$\left.+(r-1)\log\left((1+r)Q^3+(1+r)Q^2R+s\right)+\log(S^2-4Q^2)\right)=0$$

where $r=m/n\geq 1$.

Zeros of generalized Hermite polynomials

Theorem (continued)

$Q = Q(x,r)$ is the unique solution of

$$3(1+r)^2Q^4 + 8(1+r)r^{1/2}xQ^3 + 4(r-1+rx^2)Q^2 - 4 = 0$$

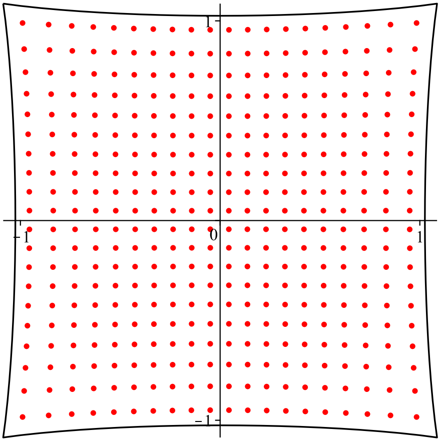
for which $Q(x,r) = -x + \mathcal{O}(x^{-2})$ as $x \to \infty$,

$$S = (1+r)Q^3 + 2r^{1/2}Q^2$$

and

$$R = -\frac{\left((1+r)^2Q^4 + 2(1+r)QS + 4\right)^{1/2}}{(1+r)Q}.$$

Zeros of generalized Hermite polynomials



Rational solutions of Painlevé IV

The Okamoto polynomials are

$$Q_n(z) = Q_{n+1,0}(z) = \mathrm{Wr}(H_{3n-1}, H_{3n-4}, \ldots, H_2),$$
$$R_n(z) = Q_{n,1}(z) = \mathrm{Wr}(H_{3n-2}, H_{3n-5}, \ldots, H_1)$$

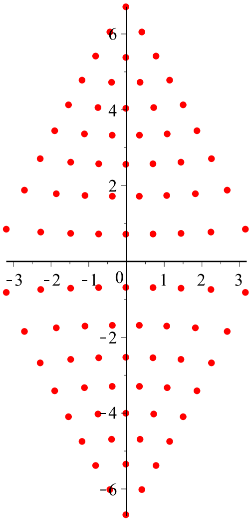
The generalized Okamoto polynomials are

$$Q_{-m,-n}(z) = H_\lambda(z),$$
$$\lambda = (m+2n, m+2n-2, \ldots, m+2, m, m, m-1, m-1, \ldots, 1, 1),$$

$$Q_{n,m+1}(z) = H_\lambda(z),$$
$$\lambda = (m+2n-1, m+2n-3, \ldots, m+1, m, m, m-1, m-1, \ldots, 1, 1).$$

$$H_\lambda(z) = \det \begin{pmatrix} H_{\lambda_1} & H_{\lambda_1+1} & H_{\lambda_1+2} & \cdots & H_{\lambda_1+k-1} \\ H_{\lambda_2-1} & H_{\lambda_2} & H_{\lambda_2+1} & \cdots & H_{\lambda_2+k-2} \\ H_{\lambda_3-2} & H_{\lambda_3-1} & H_{\lambda_3-2} & \cdots & H_{\lambda_3+k-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ H_{\lambda_k-k+1} & H_{\lambda_k-k+2} & H_{\lambda_k-k+3} & \cdots & H_{\lambda_k} \end{pmatrix}$$

Zeros of generalized Okamoto polynomials



Rational solutions of Painlevé V

$$y'' = \left(\frac{1}{2y}+\frac{1}{y-1}\right)(y')^2 - \frac{y'}{z} + \frac{(y-1)^2}{z^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{z} - \frac{y(y+1)}{2(y-1)}$$

Has rational solutions if

1. $\alpha = \frac{1}{2}(m \pm 1)^2$ and $\beta = -\frac{n^2}{2}$, with $n > 0$, $m+n$ odd, $\alpha \neq 0$ when $|m| < n$;
2. $\alpha = \frac{n^2}{2}$, $\beta = -\frac{1}{2}(m \pm 1)^2$, with $n > 0$, $m+n$ odd, $\beta \neq 0$ when $|m| < n$;
3. $\alpha = \frac{a}{2}$, $\beta = -\frac{1}{2}(a+n)^2$, and $\gamma = m$, with $m+n$ even;
4. $\alpha = (b+n)^2$, $\beta = -\frac{b^2}{2}$, and $\gamma = m$, with $m+n$ even;
5. $\alpha = \frac{1}{8}(2m+1)^2$, $\beta = -\frac{1}{8}(2n+1)^2$.

Rational solutions of Painlevé V

In terms of Umemura polynomials

$$U_{0,n}(x;\mu) = C_n \det \begin{pmatrix} L_n^r(x) & L_{n+1}^r(x) & \cdots & L_{2n-1}^r(x) \\ L_{n-2}^r(x) & L_{n-1}^r(x) & \cdots & L_{2n-3}^r(x) \\ \vdots & \vdots & \cdots & \vdots \\ L_{2-n}^r(x) & L_{3-n}^r(x) & \cdots & L_1^r(x) \end{pmatrix}, \qquad r = \mu+n-1$$

$$U_{m,0}(x;\mu) = D_m \det \begin{pmatrix} L_m^r(-x) & L_{m+1}^r(-x) & \cdots & L_{2m-1}^r(-x) \\ L_{m-2}^r(-x) & L_{m-1}^r(-x) & \cdots & L_{2m-3}^r(-x) \\ \vdots & \vdots & \cdots & \vdots \\ L_{2-m}^r(-x) & L_{3-m}^r(-x) & \cdots & L_1^r(-x) \end{pmatrix}, \quad r = \mu-m-1$$

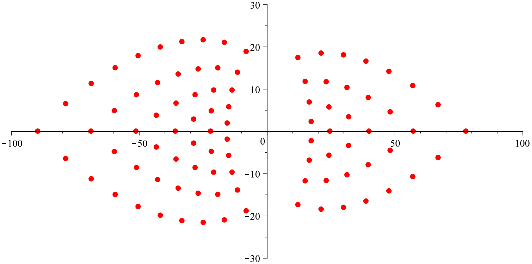
with $L_n^\alpha(x)$ the Laguerre polynomials.

and generalized Umemura polynomials $U_{m,n}(x;\mu)$

$$\det \begin{pmatrix} L_1^r(-x) & L_0^r(-x) & \cdots & L_{-m+2}^r(-x) & L_{-m+1}^r(-x) & \cdots & L_{-m-n+2}^r(-x) \\ L_3^r(-x) & L_2^r(-x) & \cdots & L_{-m+4}^r(-x) & L_{-m+3}^r(-x) & \cdots & L_{-m-n+4}^r(-x) \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ L_{2m-1}^r(-x) & L_{2m-2}^r(-x) & \cdots & L_m^r(-x) & L_{m-1}^r(-x) & \cdots & L_{m-n}^r(-x) \\ L_{n-m}^r(x) & L_{n-m+1}^r(x) & \cdots & L_{n-1}^r(x) & L_n^r(x) & \cdots & L_{2n-1}^r(x) \\ L_{n-m-2}^r(x) & L_{n-m-1}^r(x) & \cdots & L_{n-3}^r(x) & L_{n-2}^r(x) & \cdots & L_{2n-3}^r(x) \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ L_{-n-m+2}^r(x) & L_{-n-m+3}^r(x) & \cdots & L_{-n+1}^r(x) & L_{-n+2}^r(x) & \cdots & L_1^r(x) \end{pmatrix}$$

where $r = \mu - m + n - 1$.

Zeros of $U_{8,10}(\mu = 2k+1)$, $-11 \leq k \leq 9$



Rational solutions of Painlevé VI

Special function solution in terms of terminating hypergeometric functions

These are determinants with Jacobi polynomials

Little research has been done on these rational solutions.

Special function solutions of Painlevé equations

Special functions

Painlevé I	—
Painlevé II	Airy functions
Painlevé III	Bessel functions
Painlevé IV	Parabolic cylinder functions
Painlevé V	confluent hypergeometric functions
Painlevé VI	hypergeometric functions

seed function: comes from a Riccati equation

Bäcklund transformations give all the special function solutions in terms of the seed function.

Connection with orthogonal polynomials

Hankel determinant with moments of a measure μ

$$\Delta_n = \det \begin{pmatrix} m_0 & m_1 & m_2 & \cdots & m_{n-1} \\ m_1 & m_2 & m_3 & \cdots & m_n \\ m_2 & m_3 & m_4 & \cdots & m_{n+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ m_{n-1} & m_n & m_{n+1} & \cdots & m_{2n-2} \end{pmatrix}, \quad m_k = \int x^k \, d\mu(x).$$

$$m_0 = \int w(x) e^{xt} \, dx, \qquad m_k = \frac{d^k m_0}{dt^k}$$

and Δ_n is a Wronskian

$$\Delta_n = \det \frac{d^{i+j} m_0}{dt^{i+j}}.$$

Connection with orthogonal polynomials

Recurrence coefficients

$$a_n^2 = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}, \qquad b_n = \frac{\Delta_{n+1}^*}{\Delta_{n+1}} - \frac{\Delta_n^*}{\Delta_n},$$

where

$$\Delta_n^* = \det \begin{pmatrix} m_0 & m_1 & m_2 & \cdots & m_{n-2} & m_n \\ m_1 & m_2 & m_3 & \cdots & m_{n-1} & m_{n+1} \\ m_2 & m_3 & m_4 & \cdots & m_n & m_{n+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ m_{n-1} & m_n & m_{n+1} & \cdots & m_{2n-3} & m_{2n-1} \end{pmatrix} = \frac{d}{dt}\Delta_n.$$

Examples

- $w(x) = e^{-x^4+tx^2}$ $\qquad m_0 = 2^{-1/4}\sqrt{\pi}e^{t^2/8}U(0,-t/\sqrt{2})$ $\qquad \mathrm{P}_{\mathrm{IV}}$
- $w(x) = e^{-x^3/3+tx}$ $\qquad m_0 = 2\pi i \mathrm{Ai}(t)$ $\qquad \mathrm{P}_{\mathrm{II}}$
- $w_k = \frac{a^k}{(\beta)_k k!}$ on $\mathbb{N}$ $\qquad m_0 = a^{-(\beta-1)/2}\Gamma(\beta)I_{\beta-1}(2\sqrt{a})$ $\qquad \mathrm{P}_{\mathrm{III}}$
- $w(x) = x^\alpha e^{-x}e^{-s/x}$ $\qquad m_k \sim K_{-k-1}(2\sqrt{t})$ $\qquad \mathrm{P}_{\mathrm{III}}$
- $w(x) = x^{a-1}(1-x)^{b-a-1}e^{xt}$
  $m_0 = \Gamma(a)\Gamma(b-a)M(a,b,t)$ $\qquad \mathrm{P}_{\mathrm{V}}$
- $w_k = \frac{(\gamma)_k a^k}{(\beta)_k k!}$ on $\mathbb{N}$ $\qquad m_0 = M(\gamma,\beta,a)$ $\qquad \mathrm{P}_{\mathrm{V}}$
- $w(x) = x^{b-1}(1-x)^{c-b-1}(1-xt)^{-a}$
  $m_0 = \Gamma(b)\Gamma(b-c)_2F_1(a,b;c;t)$ $\qquad \mathrm{P}_{\mathrm{VI}}$

References

- P.A. Clarkson, **Painlevé equations — nonlinear special functions**, Lecture Notes in Mathematics **1883**, Springer, Berlin, 2006, pp. 331–411.
- K. Kajiwara, M. Noumi, Y. Yamada, **Geometric aspects of Painlevé equations**, J. Phys. A: Math. Theor. **50** (2017), no. 7, 073001 (164. pp.)
- W. Van Assche, **Discrete Painlevé equations for recurrence coefficients of orthogonal polynomials**, in 'Difference Equations, Special Functions and Orthogonal Polynomials', World Scientific, 2007, pp. 687–725.
- W. Van Assche, **Orthogonal Polynomials and Painlevé Equations**, Australian Mathematical Society Lecture Series, Cambridge University Press (to appear).