Gap probabilities in tiling models and Painlevé equations

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Painlevé Equations and Applications: A Workshop in Memory of A. A. Kapaev

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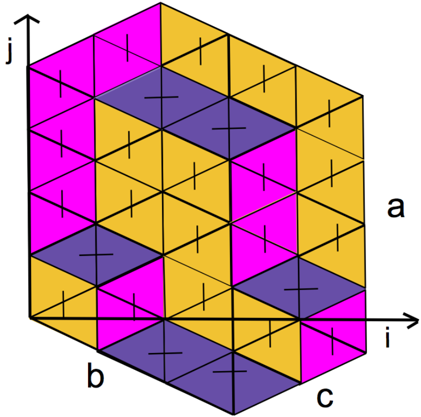
Tiling model

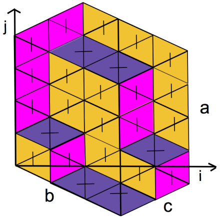


Lozenge tilings of a hexagon can be viewed as stepped surfaces.

Tiling model

Consider the probability measure on the set of tilings defined by

$$\mathcal{P}(\mathcal{T}) = \frac{\omega(\mathcal{T})}{Z(a,b,c)}, \text{ where } \omega(\mathcal{T}) = \prod_{\Diamond \in \mathcal{T}} \omega(\Diamond).$$



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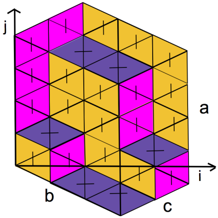

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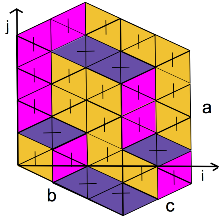

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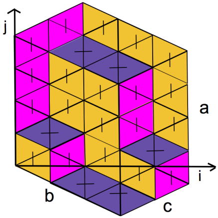

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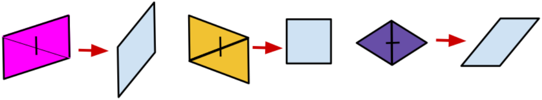
$$\omega(\mathcal{T}) = const(a,b,c) \cdot q^{-\textbf{volume}}.$$

Tiling model

Affine transformation



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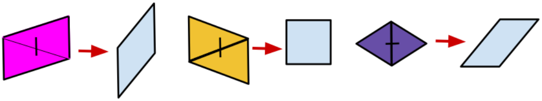

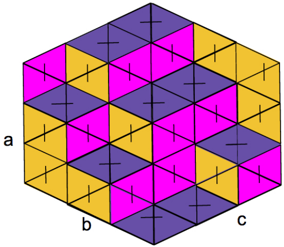

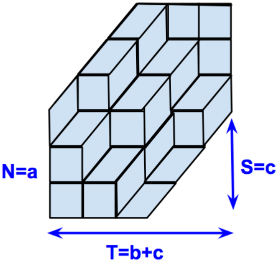

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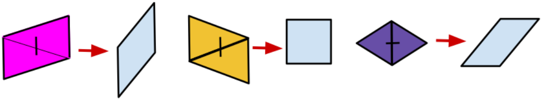
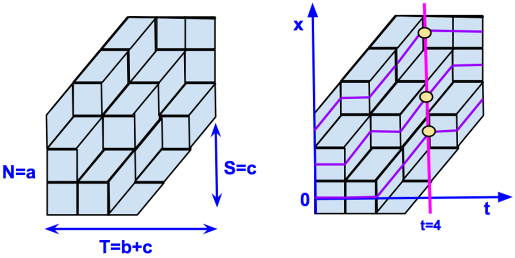
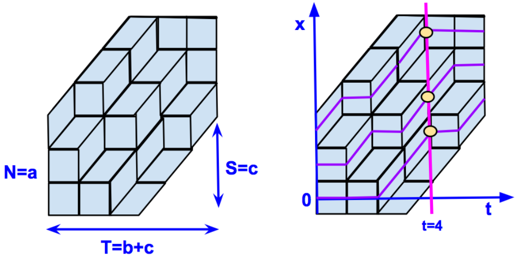


establishes a bijection between tilings and non-intersecting paths:

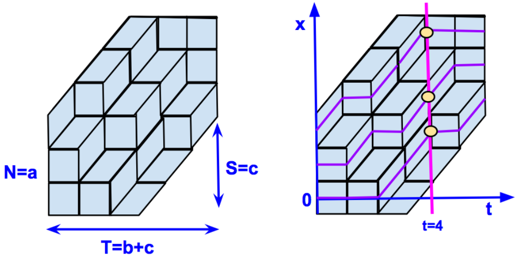


Gap probability



Fix a section t. Let the coordinates of the nodes be $C(t) = (x_1, \ldots, x_N)$.

Gap probability



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Definition

The one-interval gap probability function D_s^N is

$$D_s^N = \text{Prob}[x_i \neq s, s+1, \ldots, M] = \text{Prob}[\max\{x_i\} < s].$$

Orthogonal polynomial ensembles

Let $\mathfrak{X}$ be a discrete subset of $\mathbb{R}$. Let ω be a positive-valued function on $\mathfrak{X}$ with finite moments:

$$\sum_{x \in \mathfrak{X}} |x|^n \omega(x) < \infty, \quad n = 0, \ldots$$

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Fix $\operatorname{card}(\mathfrak{X}) > k > 0$. The orthogonal polynomial ensemble on $\mathfrak{X}$ is a probability measure on the set of all k-subsets of $\mathfrak{X}$ given by

$$\mathcal{P}(x_1, \dots, x_k) = \frac{1}{Z} \prod_{1\leqslant i<j\leqslant k} (x_i - x_j)^2 \cdot \prod_{i=1}^{k} \omega(x_i),$$

where Z is the normalizing constant.

q-Hahn

Theorem (Borodin, Gorin, Rains '2009)

$$\mathrm{Prob}\{C(t) = (x_1, \ldots, x_N)\} = const \cdot \prod_{0 \leqslant i < j \leqslant M} (q^{-x_i} - q^{-x_j})^2 \prod_{i=1}^{N} \omega_t(x_i),$$

where $\omega_t(x)$ *is the weight function of the q-Hahn polynomial ensemble up to a factor not depending on* x*.*

q-Hahn weight

Let $M \in \mathbb{Z}_{>0}$, $0 < \alpha < q^{-1}$ and $0 < \beta < q^{-1}$

$$\omega^{q}_{Hahn}(x) = (\alpha\beta q)^{-x}\frac{(\alpha q, q^{-M}; q)_x}{(q, \beta^{-1}q^{-M}; q)_x}, \text{ where}$$

$$(y_1, \ldots, y_i; q)_k = (y_1; q)_k \cdots (y_i; q)_k, \text{ and}$$

$$(y; q)_k = (1 - y) \cdots (1 - yq^{k-1}).$$

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In our case, for example

$$S - 1 < t < T - S + 1$$
$$M = S + N - 1$$
$$\alpha = -t - N$$
$$\beta = t - T - N$$

$$t < S \text{ and } T - t - S > 0$$
$$M = t + N - 1$$
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Theorem [K. '16], q-Volume case

The gap probability D_s^N for the q-Hahn ensemble can be computed recursively

$$D_s^N = \frac{(D_{s-2}^N)^2}{D_{s-1}^N} \frac{(r_{s-1}w - qva_1a_2)(r_sw - qua_1a_2)(t_{s-1} - qa_1)(t_{s-1} - qa_2)}{uva_1a_2(qa_1 - a_3)(qa_1 - a_5)(qa_2 - a_4)(qa_2 - a_6)},$$

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where the sequence (r_s, t_s) satisfies recursion

$$(r_st_{s-1} + 1)(r_{s-1}t_{s-1} + 1) = \frac{a_1a_2(t_{s-1} - a_3)(t_{s-1} - a_4)(t_{s-1} - a_5)(t_{s-1} - a_6)}{a_3a_4a_5a_6(qt_{s-1} - a_1)(qt_{s-1} - a_2)},$$

$$(r_st_s + 1)(r_st_{s-1} + 1) = \frac{uv(a_1a_2)^2(r_sa_3 + 1)(r_sa_4 + 1)(r_sa_5 + 1)(r_sa_6 + 1)}{(r_sw_s - va_1a_2)(qr_sw_s - ua_1a_2)}.$$

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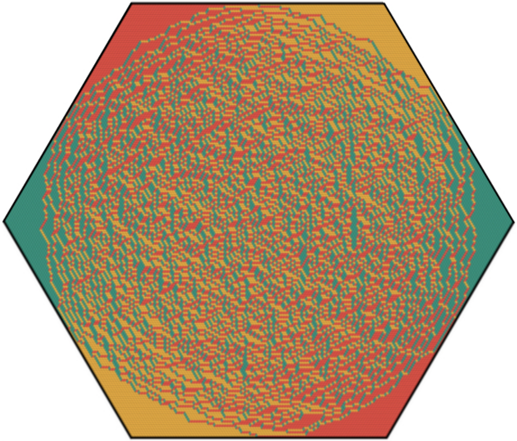
The parameters $u, v, w, a_1, \ldots, a_6$ and the initial conditions are explicitly computed in terms of α, β, q, s.

Limit shape



[Cohn–Larsen–Propp '98], [Cohn–Kenyon–Propp '01], [Kenyon–Okounkov '07].

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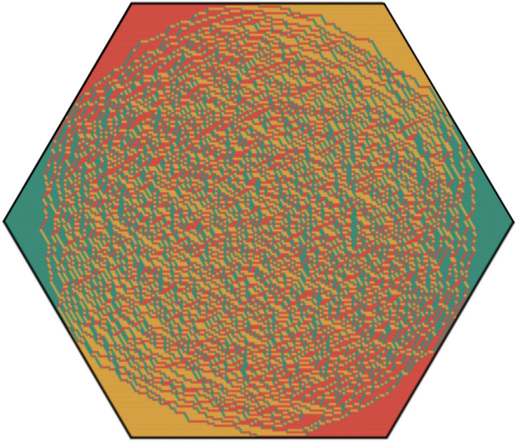

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Simulation by Leonid Petrov

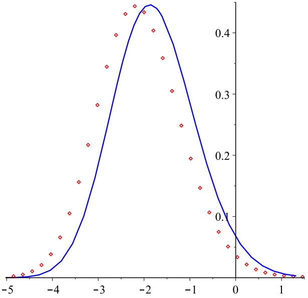
Edge fluctuations

Let all the sides linearly grow as $M \to \infty$ and $q \to 1$.

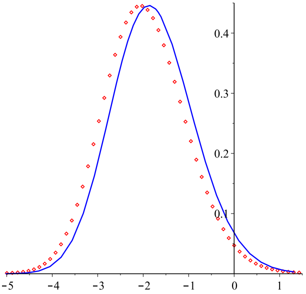
After the change of variables

$$x = c_1 M + c_2 u M^{\frac{1}{3}}$$

we present a graph of $c_2 M^{\frac{1}{3}}(D_N(x+1) - D_N(x))$, for $q = 0.99$:



$M = 2000$



$M = 10000$

Discrete RH

Theorem [Borodin-Boyarchenko 02']

Fix $\operatorname{card}(\mathfrak{X}) > k > 0$, and set $w(\psi) = \begin{bmatrix} 0 & \omega(\psi) \\ 0 & 0 \end{bmatrix}$.

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$$\text{Fix } \mathrm{card}(\mathfrak{X}) > k > 0, \text{ and set } \quad w(\psi) = \begin{bmatrix} 0 & \omega(\psi) \\ 0 & 0 \end{bmatrix}.$$

For any $s \geqslant k$ there exists unique analytic function $m_s(\psi) : \mathbb{C} \backslash \mathfrak{N}_s \to \mathrm{Mat}(\mathbb{C}, 2)$ with simple poles at points in $\mathfrak{N}_s$ such that

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$$m_s(\psi) \cdot \begin{bmatrix} \psi^{-k} & 0 \\ 0 & \psi^{k} \end{bmatrix} = \mathrm{I} + O\left(\frac{1}{\psi}\right) \text{ as } \psi \to \infty,$$

where $\mathfrak{N}_s = \{0, \ldots, s-1\}$.

Introduce matrix $A_s(z) = m_s(q^{-1}z)A_0(z)m_s^{-1}(z)$,

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In the q-Hahn case

$$\frac{q\omega^q_{Hahn}(x+1)}{\omega^q_{Hahn}(x)} = \frac{(z-\alpha q)\cdot(z-q^{-M})}{\alpha\beta(z-q)\cdot(z-\beta^{-1}q^{-M})}.$$

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- **Claim 3:** Evolution $(t_s, p_s) \to (t_{s+1}, p_{s+1})$ has the form of a discrete Painlevé equation of type $A_2^{(1)}$.

Structure of a generic $A_s(z)$

$$A(z) = \begin{bmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{bmatrix}, \quad A(0) = \begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix},$$

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We also require that

$$\det(A(z)) = uvz^6 + \mathcal{O}(z^5);$$

$$\mathrm{tr}(A(z)) = (u + v)z^3 + \mathcal{O}(z^2).$$

Evolution

$$\text{When} \quad A_s(z) \to A_{s+1}(z)$$

$$(a_1^s, a_2^s, \ldots, a_6^s, u_s, v_s, w_s) \to (a_1^{s+1}, a_2^{s+1}, \ldots, a_6^{s+1}, u_{s+1}, v_{s+1}, w_{s+1})$$

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$$\text{with } a_1^{s+1} = qa^s, a_1^{s+1} = qa^s, w_{s+1} = qw_s;$$

$$a_i^{s+1} = a_i^s \text{ for } i = 3, \ldots, 6;$$

$$u_{s+1} = u_s, v_{s+1} = v_s;$$

$$(t_s, p_s) \to (t_{s+1}, p_{s+1}) \text{ is given by } \mathrm{qP}(A_2^{(1)}).$$