Gap probabilities in tiling models and Painlevé equations

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Painlevé Equations and Applications: A Workshop in Memory of A. A. Kapaev

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Lozenge tilings of a hexagon can be viewed as stepped surfaces.
Consider the probability measure on the set of tilings defined by

\[ P(\mathcal{T}) = \frac{\omega(\mathcal{T})}{Z(a, b, c)}, \quad \text{where} \quad \omega(\mathcal{T}) = \prod_{\diamond \in \mathcal{T}} \omega(\diamond). \]
Tiling model

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Consider the probability measure on the set of tilings defined by

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\[ \omega(\mathcal{T}) = \text{const}(a, b, c) \cdot q^{-\text{volume}}. \]
Tiling model

Affine transformation
Tiling model

Affine transformation

a

b

c

N=a

S=c

T=b+c
Tiling model

Affine transformation

establishes a bijection between tilings and non-intersecting paths:
Fix a section $t$. Let the coordinates of the nodes be $C(t) = (x_1, \ldots, x_N)$.
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**Definition**

The one-interval gap probability function $D_s^N$ is

$$D_s^N = \text{Prob}[x_i \neq s, s + 1, \ldots, M] = \text{Prob}[	ext{max}\{x_i\} < s].$$
Orthogonal polynomial ensembles

Let $\mathcal{X}$ be a discrete subset of $\mathbb{R}$. Let $\omega$ be a positive-valued function on $\mathcal{X}$ with finite moments:

$$\sum_{x \in \mathcal{X}} |x|^n \omega(x) < \infty, \quad n = 0, \ldots$$
Orthogonal polynomial ensembles

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$$\sum_{x \in \mathcal{X}} |x|^n \omega(x) < \infty, \quad n = 0, \ldots$$

Fix $\text{card}(\mathcal{X}) > k > 0$. The orthogonal polynomial ensemble on $\mathcal{X}$ is a probability measure on the set of all $k$-subsets of $\mathcal{X}$ given by

$$\mathcal{P}(x_1, \ldots, x_k) = \frac{1}{Z} \prod_{1 \leq i < j \leq k} (x_i - x_j)^2 \cdot \prod_{i=1}^{k} \omega(x_i),$$

where $Z$ is the normalizing constant.
q-Hahn

Theorem (Borodin, Gorin, Rains ’2009)

\[ \text{Prob}\{C(t) = (x_1, \ldots, x_N)\} = \text{const} \cdot \prod_{0 \leq i < j \leq M} (q^{-x_i} - q^{-x_j})^2 \prod_{i=1}^{N} \omega_t(x_i), \]

where \( \omega_t(x) \) is the weight function of the q-Hahn polynomial ensemble up to a factor not depending on \( x \).
Let $M \in \mathbb{Z}_{>0}$, $0 < \alpha < q^{-1}$ and $0 < \beta < q^{-1}$

$$
\omega_{Hahn}^q(x) = (\alpha \beta q)^{-x} \frac{(\alpha q, q^{-M}; q)_x}{(q, \beta^{-1} q^{-M}; q)_x}, \text{ where }
$$

$$(y_1, \ldots, y_i; q)_k = (y_1; q)_k \cdots (y_i; q)_k, \text{ and }$$

$$(y; q)_k = (1 - y) \cdots (1 - yq^{k-1}).$$
q-Hahn weight

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\]

\[
(y; q)_k = (1 - y) \cdots (1 - yq^{k-1}).
\]

In our case, for example

\[
S - 1 < t < T - S + 1 \quad t < S \text{ and } T - t - S > 0
\]

\[
M = S + N - 1 \quad M = t + N - 1
\]

\[
\alpha = -t - N \quad \alpha = -S - N
\]

\[
\beta = t - T - N \quad \beta = S - T - N
\]
Theorem [K. ’16], q-Volume case

The gap probability $D_N^s$ for the q-Hahn ensemble can be computed recursively

$$D_N^s = \left(\frac{D_{s-2}^N}{D_{s-1}^N}\right)^2 \frac{(r_s w - qva_1 a_2)(r_s w - qua_1 a_2)(t_{s-1} - qa_1)(t_{s-1} - qa_2)}{uva_1 a_2(qa_1 - a_3)(qa_1 - a_5)(qa_2 - a_4)(qa_2 - a_6)}$$
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where the sequence $(r_s, t_s)$ satisfies recursion

$$(r_s t_{s-1} + 1)(r_s - 1 t_{s-1} + 1) = \frac{a_1 a_2(t_{s-1} - a_3)(t_{s-1} - a_4)(t_{s-1} - a_5)(t_{s-1} - a_6)}{a_3 a_4 a_5 a_6(q t_{s-1} - a_1)(q t_{s-1} - a_2)},$$

$$(r_s t_s + 1)(r_s t_{s-1} + 1) = \frac{uv(a_1 a_2)^2(r_s a_3 + 1)(r_s a_4 + 1)(r_s a_5 + 1)(r_s a_6 + 1)}{(r_s w_s - va_1 a_2)(q r_s w_s - u a_1 a_2)}.$$
Theorem [K. ’16], q-Volume case

The gap probability $D^N_s$ for the q-Hahn ensemble can be computed recursively

$$D^N_s = \frac{(D^N_{s-2})^2}{D^N_{s-1}} \frac{(r_{s-1}w - qva_1a_2)(r_sw - qua_1a_2)(t_{s-1} - qa_1)(t_{s-1} - qa_2)}{uva_1a_2(qa_1 - a_3)(qa_1 - a_5)(qa_2 - a_4)(qa_2 - a_6)},$$

where the sequence $(r_s, t_s)$ satisfies recursion

$$(r_s t_{s-1} + 1)(r_{s-1} t_{s-1} + 1) = \frac{a_1a_2(t_{s-1} - a_3)(t_{s-1} - a_4)(t_{s-1} - a_5)(t_{s-1} - a_6)}{a_3a_4a_5a_6(qt_{s-1} - a_1)(qt_{s-1} - a_2)},$$

$$(r_s t_s + 1)(r_s t_{s-1} + 1) = \frac{uv(a_1a_2)^2(r_s a_3 + 1)(r_s a_4 + 1)(r_s a_5 + 1)(r_s a_6 + 1)}{(r_sw_s - va_1a_2)(qr_sw_s - ua_1a_2)}.$$

The parameters $u, v, w, a_1, \ldots, a_6$ and the initial conditions are explicitly computed in terms of $\alpha, \beta, q, s$. 
Limit shape

[Cohn–Larsen–Propp '98], [Cohn–Kenyon–Propp '01], [Kenyon–Okounkov '07].
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Simulation by Leonid Petrov
Edge fluctuations

Let all the sides linearly grow as $M \to \infty$ and $q \to 1$.

After the change of variables

$$x = c_1 M + c_2 u M^{\frac{1}{3}}$$

we present a graph of $c_2 M^{\frac{1}{3}} (D_N(x + 1) - D_N(x))$, for $q = 0.99$:

\begin{align*}
M &= 2000 \\
M &= 10000
\end{align*}
Theorem [Borodin-Boyarchenko 02’]

Fix \( \text{card}(\mathcal{X}) > k > 0 \), and set \( w(\psi) = \begin{bmatrix} 0 & \omega(\psi) \\ 0 & 0 \end{bmatrix} \).
Discrete RH

**Theorem [Borodin-Boyarchenko 02’]**

Fix card(\(\mathcal{X}\)) > \(k\) > 0, and set \(w(\psi) = \begin{bmatrix} 0 & \omega(\psi) \\ 0 & 0 \end{bmatrix}\).

For any \(s \geq k\) there exists unique analytic function \(m_s(\psi) : \mathbb{C}\setminus\mathcal{N}_s \to \text{Mat}(\mathbb{C}, 2)\) with simple poles at points in \(\mathcal{N}_s\) such that

\[
\text{Res}_{\psi=x} m_s(\psi) = \lim_{\psi \to x} m_s(\psi) w(\psi), \ x \in \mathcal{N}_s;
\]
Theorem [Borodin-Boyarchenko 02’]

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For any $s \geq k$ there exists unique analytic function $m_s(\psi) : \mathbb{C} \setminus \mathcal{N}_s \to \text{Mat}(\mathbb{C}, 2)$ with simple poles at points in $\mathcal{N}_s$ such that
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\text{Res}_{\psi=x} m_s(\psi) = \lim_{\psi \to x} m_s(\psi) w(\psi), \quad x \in \mathcal{N}_s;
\]
\[
m_s(\psi) \cdot \begin{bmatrix} \psi^{-k} & 0 \\ 0 & \psi^k \end{bmatrix} = I + O \left( \frac{1}{\psi} \right) \quad \text{as} \ \psi \to \infty,
\]
where $\mathcal{N}_s = \{0, \ldots, s - 1\}$. 
Introduce matrix \( A_s(z) = m_s(q^{-1}z)A_0(z)m_s^{-1}(z) \),

where \( A_0(z) = \begin{bmatrix} \frac{q \omega(x+1)}{\omega(x)} & 0 \\ \omega(x) & 1 \end{bmatrix} \) with \( z = q^{-x} \).
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where

$$A_0(z) = \begin{bmatrix} \frac{q\omega(x+1)}{\omega(x)} & 0 \\ 0 & 1 \end{bmatrix} \quad \text{with} \quad z = q^{-x}.\$$

In the $q$-Hahn case

$$\frac{q\omega^q_{Hahn}(x+1)}{\omega^q_{Hahn}(x)} = \frac{(z - \alpha q) \cdot (z - q^{-M})}{\alpha \beta (z - q) \cdot (z - \beta^{-1}q^{-M})}. $$
• **Claim 1:** The gap probabilities $D_s^N$ can be computed in terms of the matrix elements of $A_s = \begin{bmatrix} a_{11}^s & a_{12}^s \\ a_{21}^s & a_{22}^s \end{bmatrix}$;
• **Claim 1:** The gap probabilities $D_s^N$ can be computed in terms of the matrix elements of $A_s = \begin{bmatrix} a_{11}^s & a_{12}^s \\ a_{21}^s & a_{22}^s \end{bmatrix}$.

• **Claim 2:** For any $s$ matrix element $a_{12}^s$ has a unique zero. Denote it by $t_s$ and $a_{22}^s[t_s]$ by $p_s$. 
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• **Claim 2:** For any $s$ matrix element $a_{12}^s$ has a unique zero. Denote it by $t_s$ and $a_{22}^s[t_s]$ by $p_s$.

• **Claim 3:** Evolution $(t_s, p_s) \rightarrow (t_{s+1}, p_{s+1})$ has the form of a discrete Painlevé equation of type $A_2^{(1)}$. 
Structure of a generic $A_s(z)$

$$A(z) = \begin{bmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{bmatrix}, \quad A(0) = \begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix},$$

$$\deg(a_{11}) \leq 3, \quad \deg(a_{12}) \leq 2, \quad \deg(a_{21}) \leq 2, \quad \deg(a_{22}) \leq 3$$
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$$\det(A(z)) = uv(z - a_1)(z - a_2)(z - a_3)(z - a_4)(z - a_5)(z - a_6).$$
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$$\det(A(z)) = uv(z - a_1)(z - a_2)(z - a_3)(z - a_4)(z - a_5)(z - a_6).$$

We also require that

$$\det(A(z)) = uvz^6 + O(z^5);$$

$$\text{tr}(A(z)) = (u + v)z^3 + O(z^2).$$
Evolution

When \( A_s(z) \rightarrow A_{s+1}(z) \)

\[
(a_1^s, a_2^s, \ldots, a_6^s, u_s, v_s, w_s) \rightarrow (a_1^{s+1}, a_2^{s+1}, \ldots, a_6^{s+1}, u_{s+1}, v_{s+1}, w_{s+1})
\]
Evolution

When \( A_s(z) \to A_{s+1}(z) \)

\[(a_1^s, a_2^s, \ldots, a_6^s, u_s, v_s, w_s) \to (a_1^{s+1}, a_2^{s+1}, \ldots, a_6^{s+1}, u_{s+1}, v_{s+1}, w_{s+1})\]

with \( a_1^{s+1} = qa^s \), \( a_1^{s+1} = qa^s \), \( w_{s+1} = qw_s \);

\( a_i^{s+1} = a_i^s \) for \( i = 3, \ldots, 6 \);

\( u_{s+1} = u_s, v_{s+1} = v_s \);

\((t_s, p_s) \to (t_{s+1}, p_{s+1})\) is given by \( qP(A_2^{(1)}) \).