Integrable gap probabilities for the Generalized Bessel process

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BESQ$^\alpha$ model

Given a $d$-dimensional Brownian motion $\{X(t), t \geq 0\}$, its squared $L^2$-norm

$$\|X(t)\|^2 = X^1(t)^2 + \ldots + X^d(t)^2$$

defines a diffusion process called **Squared Bessel Process** with parameter $\alpha = \frac{d}{2} - 1$ (BESQ$^\alpha$).

Its transition probability is given as

$$p_t^\alpha(x,y) = \frac{1}{2t} \left( \frac{y}{x} \right)^{\alpha/2} e^{-\frac{x+y}{2t}} I_\alpha \left( \frac{\sqrt{xy}}{t} \right) \quad x, y > 0$$

$$p_t^\alpha(0,y) = \frac{y^\alpha}{(2t)^{\alpha+1} \Gamma(\alpha + 1)} e^{-\frac{y}{2t}} \quad y > 0$$

**Remark**

*BESQ$^\alpha$ is well-defined for every $\alpha \in \mathbb{R}$, $\alpha > -1$.***
Consider $n$ independent copies of BESQ$^\alpha$

$$\{X_1(t), \ldots, X_n(t)\},$$

conditioned never to collide with each other. Additionally, we impose initial and final conditions to the system

$$X_j(0) = a > 0 \text{ and } X_j(1) = 0 \quad \forall j = 1, \ldots, n.$$ 

The resulting process is a determinantal point process with joint probability density

$$\frac{1}{Z_{n,t}} \det \left[ x_j^{j-1} p_t^{\alpha+1-j(\text{mod } 2)}(a, x_k) \right]_{j,k=1}^n \det \left[ x_j^{k-1} e^{-\frac{x_j}{2(T-t)}} \right]_{j,k=1}^n \, dx_1 \ldots dx_n$$

$$= \frac{1}{n!} \det \left[ K_n(x_i, x_j; t) \right]_{i,j=1}^n \, dx_1 \ldots dx_n$$

at every time $t \in (0, 1)$, where the correlation kernel $K_n$ given in terms of Multiple Orthogonal Polynomials with weights depending on the Bessel functions $I_\alpha$ (Kuijlaars et al., '09).
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Remark (Singular value distribution of the (dynamical) Laguerre Ensemble)

Let $M(t)$ be a $p \times n$ ($n > p$) matrix with i.i.d. complex Brownian entries (with mean zero and variance $2t$).

The set of singular values

$$\{\lambda_1(t), \ldots, \lambda_n(t)\}, \quad \lambda_i(t) \geq 0 \ \forall i$$

i.e. the eigenvalues of the product $M(t)^* M(t)$, has the same distribution as the above noncolliding particle system $\text{BESQ}^\alpha$ with $\alpha = 2(n - p + 1)$ (König, O’Connell, ’01).
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Introduction: the Generalized Bessel process
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Introduction: the Generalized Bessel process
Macroscopic density: for every $t \in (0, 1)$ the limiting mean density of the positions of the paths $\rho(x; t)$ exists, and is supported on the interval

$$\text{supp} (\rho(x; t)) = [p(t), q(t)] \subseteq [0, +\infty)$$

where $q(t)$ and $p(t)$ are explicit curves (loci of an algebraic equation).

Microscopic behaviour (double scaling limit):
- **Bulk:** sine kernel
- **Soft edge:** Airy kernel
- **Hard edge** (for $t > t^*$): Bessel kernel
- **Critical point** (at $t = t^*$): at a critical time $t^* = \frac{a}{a+1} \in (0, 1)$, there is a soft-to-hard transition and the local dynamics is described by the Generalized Bessel / Hard-edge Pearcey kernel.
Scaling limits

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The Generalized Bessel kernel

Theorem (Kuijlaars, Martinez-Finkelshtein, Wielonsky, ’11)

\[
\lim_{n \to +\infty} \frac{c^*}{n^{3/2}} K_n \left( \frac{c^* x}{n^{3/2}}, \frac{c^* y}{n^{3/2}}; t^* - \frac{c^* \tau}{\sqrt{n}} \right) = K_{\alpha}^{\text{crit}}(x, y; \tau) \quad x, y \in \mathbb{R}_+, \tau \in \mathbb{R},
\]

with

\[
K_{\alpha}^{\text{crit}}(x, y; \tau) = \int_{\Gamma} \frac{du}{2\pi i} \int_{\Sigma} \frac{dv}{2\pi i} e^{xu + \frac{\tau}{u} + \frac{1}{2u^2} - yv - \frac{\tau}{v} - \frac{1}{2v^2}} \frac{1}{v - u} \left( \frac{u}{v} \right)^\alpha.
\]
Remarks

Some remarks:

- the critical kernel admits an integrable representation as

\[ K_{\alpha}^{\text{crit}}(x, y; \tau) = \frac{f(x)^T g(y)}{2\pi i(x - y)} \]

\[ f(x) = \frac{1}{2\pi i} \begin{bmatrix} p(x) \\ p'(x) \\ p''(x) \end{bmatrix}, \quad g(y) = \begin{bmatrix} q''(y) - (\alpha - 2)q'(y) - \tau q(y) \\ -yq'(y) + (\alpha - 1)q(y) \\ yq(y) \end{bmatrix} \]

where \( p \) and \( q \) are solutions to suitable 3rd order ODEs.

- there exist a multi-time (matrix) version of the kernel (G.,’14 and Delvaux, Veto,’15):

\[ (K_{\alpha}^{\text{crit}})_{ij} := (H_{\alpha})_{ij} + (\Psi_{\alpha})_{\Delta_{ij}} \chi_{i<j}, \quad i,j = 1,\ldots,m; \quad \Delta_{ij} := \tau_j - \tau_i \]

\[ (H_{\alpha})_{ij}(x, y) = \int_{\Gamma} \frac{du}{2\pi i} \int_{\Sigma} \frac{d\nu}{2\pi i} \frac{4e^{xu-yv+\frac{1}{2}(\tau+\frac{1}{u}+4\Delta_{ij})^2-\frac{1}{2}(\tau+\frac{1}{v})^2}}{(v-u-4\Delta_{ij}uv)^\alpha} \left(\frac{u}{v}\right)^\alpha \]

\[ (\Psi_{\alpha})_{\Delta_{ij}}(x, y) = -\frac{1}{\Delta_{ij}} \int_{\Sigma} e^{\frac{x}{4\Delta_{ij}}} (\frac{1}{v}-1) + \frac{y}{4\Delta_{ij}} (v-1) v^{-\alpha-1} \frac{dv}{2\pi i} \]
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\]

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\]
Gap probabilities of the Generalized Bessel process

Our object of study are the gap probabilities, meaning the probability of finding no points in a given domain.

For a determinantal point process with kernel $K_n$, this boils down to calculating a Fredholm determinant:

$$P(\text{smallest point } > s) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{[0,s]^k} \det [K_n(x_i, x_j)]_{i,j=1,\ldots,k} \, dx_1 \ldots dx_k$$

$$= \det \left( \operatorname{Id}_{L^2(\mathbb{R}_+)} - K_n \bigg|_{[0,s]} \right)$$

and in the scaling limit regime

$$\det \left( \operatorname{Id}_{L^2(\mathbb{R}_+)} - K_n \bigg|_{\left[0, \frac{c^* s}{n^{3/2}} \right]} \right) \to \det \left( \operatorname{Id}_{L^2(\mathbb{R}_+)} - K_{\alpha}^{\text{crit}} \bigg|_{[0,s]} \right) \quad \text{as } n \uparrow +\infty.$$
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Differential identity

Theorem (G., ’14)

Let $s > 0$ and $K^\text{crit}_\alpha$ be the integral operator acting on $L^2(\mathbb{R}_+)$ with kernel defined above. Then, the following differential formula for gap probabilities holds

\[
d_{s,\tau} \ln \det \left( \text{Id}_{L^2(\mathbb{R}_+)} - K^\text{crit}_\alpha \bigg|_{[0,s]} \right) = (Y_1)_{2,2} ds - \left( \hat{Y}_0^{-1}\hat{Y}_1 \right)_{2,2} d\tau
\]

where $Y$ is the solution to a suitable RH problem and $Y_1$ and $\hat{Y}_j$ are the coefficients appearing in the asymptotic expansion of $Y$ at infinity and at zero, respectively.
The Riemann-Hilbert problem for $Y$

Find a $2 \times 2$ matrix-valued function $Y = Y(\lambda; s, \tau)$ such that

- $Y$ is analytic on $\mathbb{C} \setminus (\Gamma \cup \Sigma)$
- $Y$ admits a limit when approaching the contours from the left $Y_+$ or from the right $Y_-$ (according to their orientation), and the following jump condition holds

$$J_\Sigma = \begin{bmatrix} 1 & -\lambda - \alpha e^{-\lambda s} - \frac{1}{2\lambda^2} \\ \lambda - \frac{1}{2\lambda^2} & 0 \end{bmatrix}$$

$$J_\Gamma = \begin{bmatrix} 1 & 0 \\ -\lambda^2 e^\lambda s + \frac{1}{2\lambda^2} & 1 \end{bmatrix}$$

- $Y$ has the following (normalized) behaviour at $\infty$:

$$Y(\lambda) = I + \frac{Y_1(s, \tau)}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \quad \lambda \to \infty.$$
Proposition (part I)

The following identity holds

\[
\det \left( \text{Id}_{L^2(\mathbb{R}_+)} - K^{\text{crit}}_{\alpha} \big|_{[0,s]} \right) = \det \left( \text{Id}_{L^2(\Sigma \cup \Gamma)} - H \right)
\]

where \( H \) is an Its-Izergin-Korepin-Slavnov ('90) integral operator with kernel

\[
H = \frac{f(\lambda)^T g(\mu)}{\lambda - \mu}
\]

\[
f(\lambda) = \frac{1}{2\pi i} \begin{bmatrix} e^{-\frac{\lambda s}{2}} \chi_\Sigma(\lambda) \\ \chi_\Gamma(\lambda) \end{bmatrix} \quad g(\mu) = \begin{bmatrix} \mu^\alpha e^{\frac{\mu s}{2} + \frac{\tau}{\mu} + \frac{1}{2\mu^2}} \chi_\Gamma(\mu) \\ \mu^{-\alpha} e^{-\frac{\mu s}{2} - \frac{\tau}{\mu} - \frac{1}{2\mu^2}} \chi_\Sigma(\mu) \end{bmatrix}.
\]

The result can be proved by noticing that \( K^{\text{crit}}_{\alpha} \big|_{[0,s]} \) is unitarily equivalent (via Fourier transform) to an integral operator that can be decomposed as the above operator \( H \).
IIKS operators naturally carry an associated RH problem, whose solution $Y$ is tied to the invertibility of their resolvent operator.

Given such RH problem, we make use of a major (and more general) result due to Bertola (’10) and Bertola-Cafasso (’11) which, if applied to our case, reads as follows

**Proposition (part II)**

Define the quantity for $\rho = s, \tau$

$$\omega(\partial_\rho) := \int_{\Sigma \cup \Gamma} \text{Tr} \left[ Y^{-1} Y' \left( \partial_\rho J \right) J^{-1} \right] \frac{d\lambda}{2\pi i}.$$  

Then, we have the equality

$$\omega(\partial_\rho) = \partial_\rho \ln \det \left( \text{Id}_{L^2(\Sigma \cup \Gamma)} - \mathbb{H} \right).$$  

By expanding the solution $Y$ at infinity and at zero, this identity can be further simplified and explicitly calculated and it yields the final result:

$$d_{s, \tau} \ln \det \left( \text{Id}_{L^2(\Sigma \cup \Gamma)} - \mathbb{H} \right) = \left( Y_1 \right)_{2, 2} ds - \left( \hat{Y}_0^{-1} \hat{Y}_1 \right)_{2, 2} d\tau.$$
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$$d_{s,\tau} \ln \det \left( \text{Id}_{L^2(\Sigma \cup \Gamma)} - \mathbb{H} \right) = (Y_1)_{2,2} \, ds - (\hat{Y}_0^{-1} \hat{Y}_1)_{2,2} \, d\tau.$$
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**Proposition (part II)**

Define the quantity for $\rho = s, \tau$

$$
\omega(\partial \rho) := \int_{\Sigma \cup \Gamma} \text{Tr} \left[ Y_{-1}^{-1} Y_{-1}' (\partial \rho J) J^{-1} \right] \frac{d\lambda}{2\pi i}.
$$

Then, we have the equality

$$
\omega(\partial \rho) = \partial \rho \ln \det \left( \text{Id} L^2(\Sigma \cup \Gamma) - H \right).
$$

By expanding the solution $Y$ at infinity and at zero, this identity can be further simplified and explicitly calculated and it yields the final result:

$$
ds,\tau \ln \det \left( \text{Id} L^2(\Sigma \cup \Gamma) - H \right) = (Y_{1})_{2,2} \, ds - \left( \hat{Y}_0^{-1} \hat{Y}_1 \right)_{2,2} \, d\tau.
$$
A few words on $\omega(\partial)$

The solution to the RH problem $Y$ solves a rational ODE (up to a gauge transformation)

$$\frac{dY}{d\lambda} = A(\lambda)Y(\lambda)$$

With this extra property, it turns out that given the Malgrange-Bertola differential form (Bertola, ’10)

$$\omega(\partial) = \int_{\Sigma \cup \Gamma} \text{Tr} \left[ Y^{-1}Y' (\partial J) J^{-1} \right] \frac{d\lambda}{2\pi i},$$

then $\omega$ is the logarithmic total differential of the isomonodromic $\tau$ function:

$$d\omega = 0 \quad \text{and} \quad e^{\int \omega} = \tau_{\text{JMU}}.$$

Conclusion

*We give a specific geometrical meaning to a probabilistic quantity:*

$$\tau_{\text{JMU}} = \det \left( \text{Id}_{L^2(\mathbb{R}_+)} - K_{\alpha}^{\text{crit}} \right) \bigg|_{[0,s]} = \begin{cases} \text{infinitesimal fluctuation of smallest path of BESQ}^{\alpha} \\ \text{at the critical time } t^* \end{cases}$$

*(up to a normalization constant).*
Some more remarks

This method for studying gap probabilities was first introduced by Bertola-Cafasso (’11) to study the well-known Airy and Pearcey processes.

It has been successfully applied later on for other processes: Bessel process (G., ’14), tacnode process (G., ’14), hard-edge processes for product of Ginibre or truncated unitary matrices and Muttalib-Borodin process (Claeys, G., Stivigny, ’17).

The key point is that the restriction of the given operator to an interval $K|_I$ is isometrically equivalent to an IIKS operator. The main clue is the double-contour integral representation of the type:

$$K(x, y) = \int_{\Sigma_1} \frac{du}{2\pi i} \int_{\Sigma_2} \frac{dv}{2\pi i} \frac{F(u; x)F^{-1}(v; y)}{u - v}$$
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$$K(x, y) = \int_{\Sigma_1} \frac{du}{2\pi i} \int_{\Sigma_2} \frac{dv}{2\pi i} \frac{\mathcal{F}(u; x)\mathcal{F}^{-1}(v; y)}{u - v}$$
Given
\[ ds, \tau \ln \det \left( \text{Id}_{L^2(\Sigma \cup \Gamma)} - \mathbb{H} \right) = (Y_1)_{2,2} \, ds - \left( \tilde{Y}_0^{-1} \tilde{Y}_1 \right)_{2,2} \, d\tau \]
we can further study our RH problem to draw some interesting conclusions:

- asymptotic behaviour of gap probability (large/small gap, degeneration regimes) \( \rightarrow \) Deift-Zhou steepest descent method

- integrability and differential equations (Tracy-Widom) \( \rightarrow \) Lax pair, hamiltonian formalism
Given
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The Lax triplet

From the RH problem $Y$ associated to our critical kernel $K_{\alpha}^{\text{crit}}$

$$Y_{+}(\lambda) = Y_{-}(\lambda) \begin{cases} 
\begin{bmatrix} 
1 & -\lambda^{-\alpha}e^{-\lambda s - \frac{\tau}{\lambda} - \frac{1}{2\lambda^2}} \\
0 & 1 \\
-\lambda^{-\alpha}e^{\lambda s + \frac{\tau}{\lambda} + \frac{1}{2\lambda^2}} & 1 
\end{bmatrix} & \lambda \in \Sigma \\
\begin{bmatrix} 
1 \\
0 \\
-\lambda^{-\alpha}e^{\lambda s + \frac{\tau}{\lambda} + \frac{1}{2\lambda^2}} 
\end{bmatrix} & \lambda \in \Gamma 
\end{cases}$$

we can derive the following Lax triplet:

$$A = A^{(\lambda)} = A_{0} + \frac{A_{-1}}{\lambda} + \frac{A_{-2}}{\lambda^2} + \frac{A_{-3}}{\lambda^3},$$

$$B = B^{(s)} = \lambda B_{1} + B_{0},$$

$$C = C^{(\tau)} = \frac{C_{-1}}{\lambda}. $$
Up to a change of variables $\lambda \mapsto \frac{1}{\lambda}$, the matrices $\{A, C\}$ is

$$A = \frac{\lambda}{2} \sigma_3 + A_0 + \frac{A_{-1}}{\lambda} + \frac{A_{-2}}{\lambda^2}, \quad C = \frac{\lambda}{2} \sigma_3 + C_0$$

with coefficients

$$A_0 = \begin{bmatrix} -\frac{1}{w} \left[ v_{\tau} + u \left( v^2 - \Theta \right) \right] & \frac{uw}{2} \end{bmatrix}, \quad A_{-2} = \begin{bmatrix} -\frac{1}{w} \left( v^2 - \Theta \right) & w \end{bmatrix},$$

$$A_{-1} = \begin{bmatrix} u \left[ v_{\tau} + u \left( v^2 - \Theta \right) \right] + \frac{\alpha}{2} & w \left[ u_{\tau} - 2u^2v + \tau u \right] \\ \frac{1}{w} \left[ \left( u_{\tau} - 4u^2v + \tau u \right) \left( v^2 - \Theta \right) - 2uvv_{\tau} - \alpha v + \tilde{\Theta} \right] & -u \left[ v_{\tau} + u \left( v^2 - \Theta \right) \right] - \frac{\alpha}{2} \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 0 & uw \\ -\frac{1}{w} \left[ v_{\tau} + u \left( v^2 - \Theta \right) \right] & 0 \end{bmatrix}.$$

where $u, v, w$ depend on both the variables $s$ and $\tau$ and suitable constants $\Theta, \tilde{\Theta}$. 
Theorem (G., Cafasso)

The Lax pair \( \{A, C\} \) is the Lax pair associated to the second member of the Painlevé III hierarchy defined by Sakka ('09).

The compatibility equation (for the \((\lambda, \tau)\) variables) yields the system

\[
\begin{align*}
  u_{\tau\tau} &= (6uv - \tau)u_\tau - 6u^3v^2 + 2\tau u^2v + 2\Theta u^3 - (\alpha + 1)u + 1 \\
  v_{\tau\tau} &= -(6uv - \tau)v_\tau - 2u(3uv - \tau)(v^2 - \Theta) - \alpha v + \tilde{\Theta}.
\end{align*}
\]

which can be further reduced to a 4\(^{th}\)-order equation for the function \(u\).
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The quest for a Garnier system...

As in the classical Painlevé theory (Jimbo, Miwa, Ueno, '81), we would like to find a completely integrable (Hamiltonian) system associated with the Lax triplet \( \{A, B, C\} \).

In this case, we have two independent parameters that describe the flow, the time \( \tau \) and the space \( s \), therefore we need a 2D version of Hamiltonian system (Garnier system, '26) for the canonical coordinates \((\mu_1, \mu_2; \lambda_1, \lambda_2)\):

\[
\begin{align*}
\frac{\partial \lambda_j}{\partial \tau} &= \frac{\partial H_\tau}{\partial \mu_j} \\
\frac{\partial \mu_j}{\partial \tau} &= -\frac{\partial H_\tau}{\partial \lambda_j} \\
\frac{\partial \lambda_j}{\partial s} &= \frac{\partial H_s}{\partial \mu_j} \\
\frac{\partial \mu_j}{\partial s} &= -\frac{\partial H_s}{\partial \lambda_j}
\end{align*}
\]

with rational Hamiltonians \( H_\tau = H_t(\lambda_j, \mu_j; s, \tau) \) and \( H_s = H_s(\lambda_j, \mu_j; s, \tau) \).
The quest for a Garnier system...

As in the classical Painlevé theory (Jimbo, Miwa, Ueno, ’81), we would like to find a completely integrable (Hamiltonian) system associated with the Lax triplet \( \{A, B, C\} \).

In this case, we have two independent parameters that describe the flow, the time \( \tau \) and the space \( s \), therefore we need a 2D version of Hamiltonian system (Garnier system, ’26) for the canonical coordinates \((\mu_1, \mu_2; \lambda_1, \lambda_2)\):

\[
\begin{align*}
\frac{\partial \lambda_j}{\partial \tau} &= \frac{\partial H_\tau}{\partial \mu_j} \\
\frac{\partial \mu_j}{\partial \tau} &= -\frac{\partial H_\tau}{\partial \lambda_j} \\
\frac{\partial \lambda_j}{\partial s} &= \frac{\partial H_s}{\partial \mu_j} \\
\frac{\partial \mu_j}{\partial s} &= -\frac{\partial H_s}{\partial \lambda_j}
\end{align*}
\]

with rational Hamiltonians \( H_\tau = H_t(\lambda_j, \mu_j; s, \tau) \) and \( H_s = H_s(\lambda_j, \mu_j; s, \tau) \).

\[
K(1 + 2 + 2) \quad K(2 + 3) \\
K(1 + 1 + 1 + 1 + 1) \quad K(1 + 1 + 1 + 2) \quad K(5) \quad K\left(\frac{9}{2}\right) \\
K(1 + 1 + 3) \quad K(1 + 4)
\]
Action plan

**Step 1:** we identify the canonical coordinates in our system

\[ \{\lambda_j\}_{j=1,2} \] as the solutions of the equation \( (A(\lambda; s, \tau))_{1,2} = 0 \)

\[ \{\mu_j\}_{j=1,2} \] as \( \mu_j = (A(\lambda_j; s, \tau))_{1,1} \)

**Step 2:** the compatibility equations of the Lax triplet yield a system of 8 differential equations (4 for the variable \( s \), 4 for the variable \( \tau \)) which can be represented as a Garnier system

\[
\begin{align*}
\frac{\partial \lambda_j}{\partial \tau} &= \frac{\partial H_\tau}{\partial \mu_j} \\
\frac{\partial \mu_j}{\partial \tau} &= -\frac{\partial H_\tau}{\partial \lambda_j}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial \lambda_j}{\partial s} &= \frac{\partial H_s}{\partial \mu_j} \\
\frac{\partial \mu_j}{\partial s} &= -\frac{\partial H_s}{\partial \lambda_j}
\end{align*}
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with rational Hamiltonians \( H_\tau = H_\tau(\lambda_j, \mu_j; s, \tau) \) and \( H_s = H_s(\lambda_j, \mu_j; s, \tau) \).
Integrable gap probabilities for the Generalized Bessel process
Painlevé and hamiltonian connection
Garnier system

\[ H_\tau = -\frac{\lambda_1^2 \mu_1^2}{\lambda_1 - \lambda_2} + \frac{\lambda_2^2 \mu_2^2}{\lambda_1 - \lambda_2} - \frac{s^2 (\lambda_1 + \lambda_2)}{4\lambda_1^2 \lambda_2^2} + \frac{\tau^2 (\lambda_1 + \lambda_2)}{4} - \frac{k s}{\lambda_1 \lambda_2} \]

\[ - \frac{\tau (\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2)}{2} + \frac{\lambda_1^3}{4} + \frac{\lambda_1^2 \lambda_2}{4} + \frac{\lambda_1^2 \lambda_2}{4} + \frac{\lambda_2^3}{4} - \frac{(\alpha + 1) \lambda_1 + 2\alpha \lambda_2}{2} \]

\[ H_s = -\frac{\lambda_1 \lambda_2 (\lambda_1 \mu_1^2 + \mu_1)}{s (\lambda_1 - \lambda_2)} + \frac{\lambda_1 \lambda_2 (\lambda_2 \mu_2^2 + \mu_2)}{s (\lambda_1 - \lambda_2)} + \frac{\tau^2 \lambda_1 \lambda_2}{4s} - \frac{k (\lambda_1 + \lambda_2)}{\lambda_1 \lambda_2} - \frac{\alpha \lambda_1 \lambda_2}{2s} \]

\[ - \frac{s (\lambda_1 + \lambda_2)}{4\lambda_1^2 \lambda_2} - \frac{\tau \lambda_2 (\lambda_1^2 + \lambda_1 \lambda_2)}{2s} + \frac{\lambda_1 \lambda_2 (\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2 - 2)}{4s} - \frac{s}{4\lambda_2^2} \]

**Remark**

These Hamiltonians are different from the Hamiltonians of the $K(2 + 3)$ system defined in Okamoto-Kimura, ’86. The identification process is on-going...
1 Introduction: the Generalized Bessel process

2 First result: differential identity for gap probabilities
   • Sketch of the proof

3 Painlevé and hamiltonian connection (joint with M. Cafasso, U. Angers)
   • Painlevé-type equation
   • Garnier system

4 Conclusive remarks and open questions
Explicit connection between Hamiltonians and gap probabilities/RH problem for $K_\alpha^{\text{crit}}$?

$$d_{s,\tau} \ln \det \left( \text{Id}_{L^2(R_+)} - K_\alpha^{\text{crit}} \bigg|_{[0,s]} \right) = \mathcal{L}_1 (H_\tau, H_s) \, ds + \mathcal{L}_2 (H_\tau, H_s) \, d\tau$$
New horizons

2D Painlevé-Calogero correspondence? Quantization?

Reduce the Hamiltonians into a mechanical system type of the form
\[ H(\vec{p}, \vec{q}) = ||\vec{p}||^2 + V(\vec{q}). \]

Via standard quantization, study the Schrödinger system

\[
\hbar \frac{\partial}{\partial \tau} \Phi(x; s, \tau) = \hat{H}_\tau \left( x_j, \hbar \frac{\partial}{\partial x_j}; s, \tau \right) \Phi(x; s, \tau)
\]

\[
\hbar \frac{\partial}{\partial s} \Phi(x; s, \tau) = \hat{H}_s \left( x_j, \hbar \frac{\partial}{\partial x_j}; s, \tau \right) \Phi(x; s, \tau)
\]
Further work:

- what will the Lax pair \{A, B\} yield?

\[
A = A_0 + \frac{A_{-1}}{\lambda} + \frac{A_{-2}}{\lambda^2} + \frac{A_{-3}}{\lambda^3}, \quad B = \lambda B_1 + B_0;
\]

- asymptotic behaviour?

**Conjecture**: degeneration of the gap probabilities of \(K_{\alpha}^{\text{crit}}\) into gap probabilities of the Airy process (for \(\tau \nearrow +\infty\)) or the Bessel process (for \(\tau \searrow -\infty\)).
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References


Thanks for your attention!