Geometry of Discrete Painlevé Equations and Applications

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Based on the joint work with *Tomoyuki Takenawa*
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Painlevé Equations and Applications:
A Workshop in Memory of A. A. Kapaev

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The University of Michigan

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Prologue

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$$\begin{cases} \bar{x} = \dfrac{(\alpha - \beta)(\alpha x(\theta_1^1 - \theta_1^2) + (1 + \theta_0^2)(x(y - \theta_1^2) + y(\theta_0^1 - \theta_0^2)))}{(\alpha - \beta)(x(y - \theta_1^2) + (\theta_0^1 - \theta_0^2)y) - \alpha(\theta_1^1 + 1)(\theta_0^1 - \theta_0^2)} \\ \bar{y} = \dfrac{(\alpha - \beta)(y(x + \theta_0^1 - \theta_0^2) - \theta_1^2 x)}{\alpha(\theta_0^1 - \theta_0^2)} \end{cases}, \qquad (1)$$

where θ_i^j and κ_i are some parameters and

$$\alpha(x,y) = \frac{\left(y r_1 + \frac{x(\theta_0^2 r_1 + r_2)}{x + \theta_0^1 - \theta_0^2}\right)}{(x+y)(\theta_1^1 - \theta_1^2)}, \qquad \beta(x,y) = \frac{((y + \theta_0^2) r_1 + r_2)}{(x+y)(\theta_1^1 - \theta_1^2)},$$

$$r_1(x,y) = \kappa_1\kappa_2 + \kappa_2\kappa_3 + \kappa_3\kappa_1 - (y - \theta_1^2)(x - \theta_0^2) - \theta_0^1(y + \theta_0^2) - \theta_1^1(\theta_0^1 + \theta_0^2 + \theta_1^2),$$

$$r_2(x,y) = \kappa_1\kappa_2\kappa_3 + \theta_1^1((y - \theta_1^2)(x - \theta_0^2) + \theta_0^1(y + \theta_0^2)).$$

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$$r_1(x,y) = \kappa_1\kappa_2+\kappa_2\kappa_3+\kappa_3\kappa_1-(y-\theta_1^2)(x-\theta_0^2)-\theta_0^1(y+\theta_0^2)-\theta_1^1(\theta_0^1+\theta_0^2+\theta_1^2),$$

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$$\begin{cases} (f+g)(\bar{f}+g) = \dfrac{(g+b_1)(g+b_2)(g+b_3)(g+b_4)}{(g-b_5-\delta)(g-b_6-\delta)} \\ (\bar{f}+g)(\bar{f}+\bar{g}) = \dfrac{(\bar{f}-b_1)(\bar{f}-b_2)(\bar{f}-b_3)(\bar{f}-b_4)}{(\bar{f}+b_7-\delta)(\bar{f}-b_8-\delta)} \end{cases}, \tag{2}$$

where $b_1,\dots,b_8$ are some parameters and $\delta = b_1+\cdots+b_8$.

Both equations are in fact very natural expressions (in their respective settings, of course) of difference Painlevé equations of type d-$P\left(A_2^{(1)*}\right)$ with symmetry $\widetilde{W}\left(E_6^{(1)}\right)$, and so a question about the relationship between the them is a very reasonable one.

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According to the Sakai’s classification scheme, a *discrete Painlevé equation* is a birational map of a complex projective plane that corresponds to a *translation element* in the symmetry sub-lattice of a Picard lattice of a certain rational algebraic surface, known as the *Okamoto Space of Initial Conditions*, that is obtained when we resolve the indeterminacies of the equation by using a blowup procedure. Our approach is to exploit the structure of the extended affine Weyl symmetry group $\widetilde{W}\left(E_6^{(1)}\right)$ of the surface.

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Main result: These two equations are *equivalent* through an explicit change of variables transforming one equation into the other:

$$f = \frac{x(y-\theta_1^1) + y(\theta_0^1+\kappa_1) + (\theta_0^2+\kappa_1)(\theta_0^1+\theta_0^2+\theta_1^1+2\kappa_1)}{y+\theta_0^2+\kappa_1}$$

$$g = \frac{x(y-\theta_0^2-\theta_1^1-\kappa_1) + y(\theta_0^1-\theta_0^2) + (\theta_0^2+\kappa_1)(\theta_0^1+\theta_0^2+2\kappa_1)}{x-\theta_0^2-\kappa_1}$$

Classification Scheme for Painlevé Equations

n=1: L. Fuchs, H. Poincaré

- $\left(\dfrac{dy}{dt}\right)^2 = 4y^3 - g_2 y - g_3, \qquad g_2, g_3 \in \mathbb{C}$ Weierstrass $\wp(t|g_2, g_3)$
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(P-I) $\frac{d^2y}{dt^2} = 6y^2 + t;$

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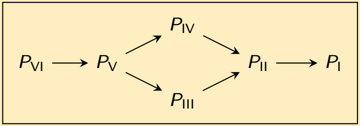

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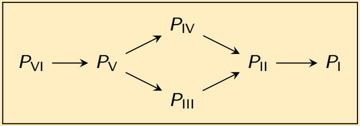

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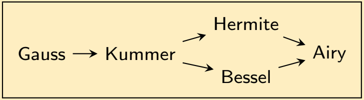

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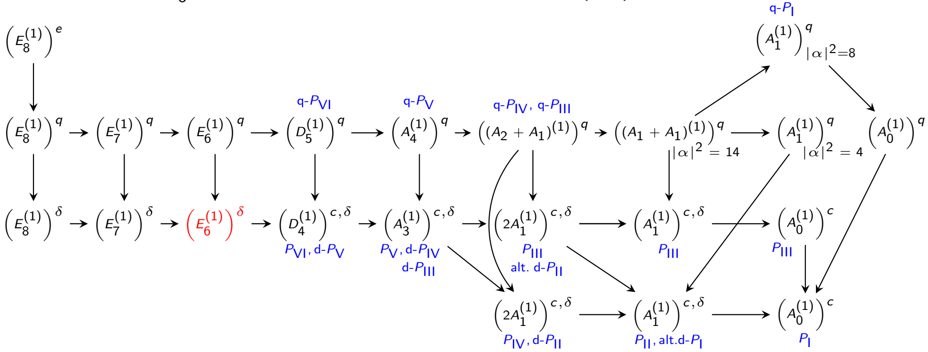

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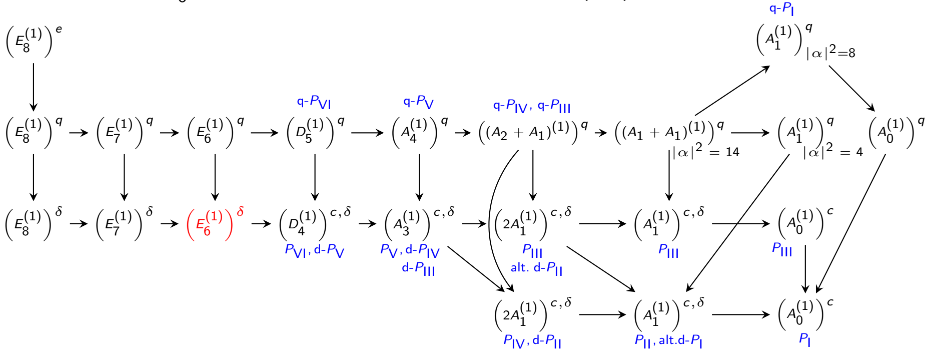

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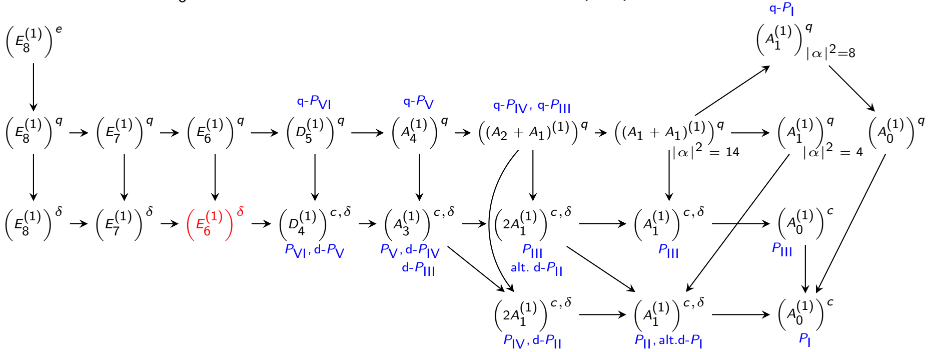

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One of my goals for today is to explain the main ideas behind this scheme.

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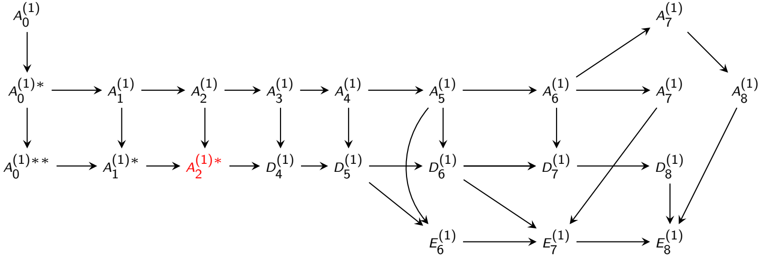

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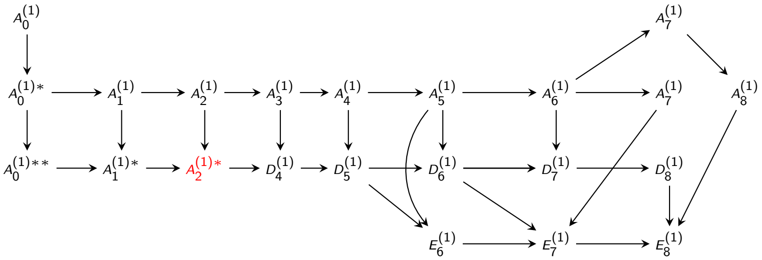

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In this description each letter stands for a Dynkin diagram describing intersection configuration of the irreducible components of an anti-canonical divisor of a certain algebraic surface, known as the *generalized Halphen surface*, that is obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at *eight* (possibly infinitely-close) points.

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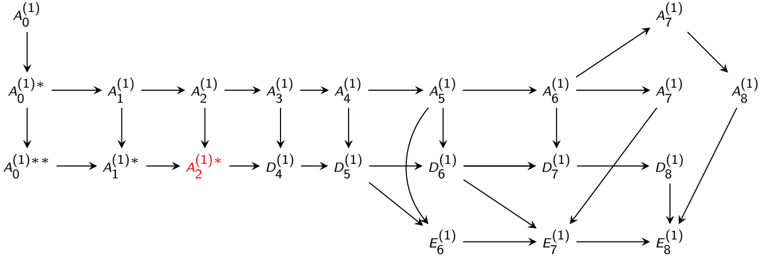

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The very top $A_0^{(1)}$ node corresponds to the the points lying on a smooth elliptic curves, $A_0^{(1)*}$ is a nodal curve (hence the multiplicative dynamic on the parameters), $A_0^{(1)**}$ is a cusp curve (hence the additive dynamic on the parameters), and others are various degenerations into rational components.

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For the geometric description, the most natural approach is via the deautonomization of QRT maps, so we briefly recall this construction.

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$$a_{00}x^2y^2 + a_{01}x^2y + a_{02}x^2 + a_{10}xy^2 + a_{11}xy + a_{12}x + a_{20}y^2 + a_{21}y + a_{22} = 0.$$

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- This equation can be written as

$$\mathbf{x}^T\mathbf{A}\mathbf{y} = \begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y^2 \\ y \\ 1 \end{bmatrix} = \sum_{i,j=0}^{2} a_{ij}x^{2-i}y^{2-j} = 0,$$

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$$\mathbf{x} = \langle x^2, x, 1 \rangle, \quad \mathbf{y} = \langle y^2, y, 1 \rangle, \qquad \mathbf{A} \in \mathrm{Mat}_{3\times 3}(\mathbb{C}).$$

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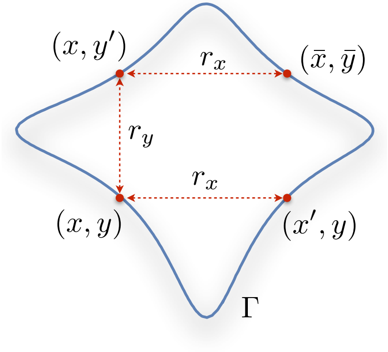
- In general, Γ is an *elliptic curve* that can be rewritten in a Weierstrass normal form $Y^2 = 4X^3 - g_2X - g_3$.

Since Γ has bi-degree $(2,2)$, we can define two *involutions*,

$$r_x : (x,y) \to (x',y) \text{ and } r_y : (x,y) \to (x,y')$$



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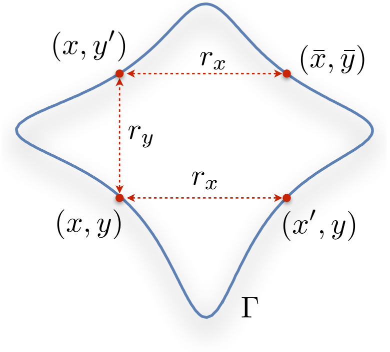

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The main idea of the QRT map is to extend $r_x \circ r_y$ to all of the $\mathbb{P}^1 \times \mathbb{P}^1$.

Geometry of a QRT Map

For that, take *two* matrices $\mathbf{A}, \mathbf{B} \in \mathrm{Mat}_{3\times 3}(\mathbb{C})$ and consider a *pencil* (i.e., a one-dimensional family) of such curves

$$\Gamma_{[\alpha:\beta]}: \qquad \alpha \mathbf{x}^T \mathbf{A} \mathbf{y} + \beta \mathbf{x}^T \mathbf{B} \mathbf{y} = 0, \quad [\alpha:\beta] \in \mathbb{P}^1$$

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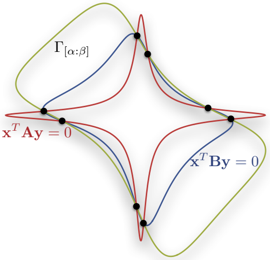

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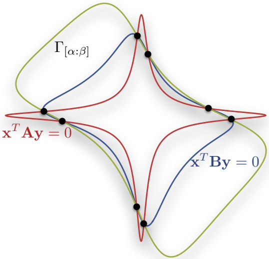

Then, given a point (x_*, y_*), there is only one curve from a family with the parameter $[\alpha : \beta] = [-\mathbf{x}_*^T \mathbf{B} \mathbf{y}_*, \mathbf{x}_*^T \mathbf{A} \mathbf{y}_*]$, *except* for the **eight** base points $\mathbf{x}_*^T \mathbf{A} \mathbf{y}_* = \mathbf{x}_*^T \mathbf{B} \mathbf{y}_* = 0$.

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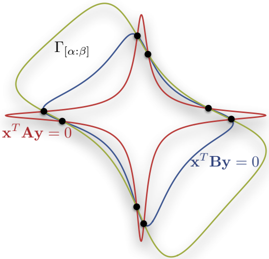

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Resolving these points using the blowup, we get a rational elliptic surface $\mathcal{X}$ with the *QRT* automorphism $r_x \circ r_y$ preserving the elliptic fibration $\pi : \mathcal{X} \to \mathbb{P}^1$, and $\pi^{-1}([\alpha:\beta])$ is an elliptic curve except for 12 points corresponding to *singular fibers* (classified by K. Kodaira into 22 types).

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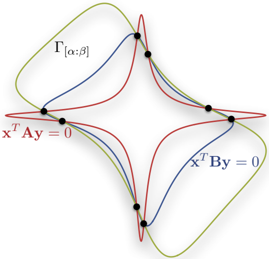

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Deautonomization is performed with different choices of a fiber on which the blowup points lie (that fiber is exactly the anti-canonical divisor, $-K_{\mathcal{X}}$ that is the key object of the geometric theory). Allowing the points move along a particular fiber, either smooth or singular, breaks down the elliptic surface structure and the dynamic becomes non-autonomus.

The $A_2^{(1)*}$ Deautonomization Example

For the example that we are interested in, we can take the matrices **A**, **B** as

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & -(a + a^{-1}) & 1 \\ -(a + a^{-1}) & (b + b^{-1})^2 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

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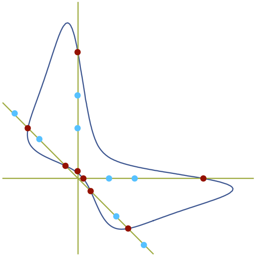

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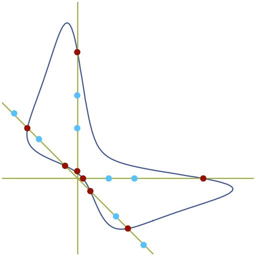

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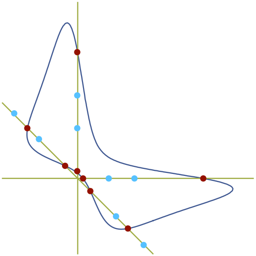

The base points of the map are shown in red, allowing them to move along the $A_2^{(1)*}$-fiber (points shown in blue) resulted in the deautonomization example of Grammaticos-Ramani-Ohta. However, we can in fact create the mapping starting just from the Dynkin diagrams and a choice of a translation element. We explain how to do that next.

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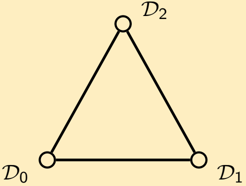
Dynkin diagram $A_2^{(1)}$ and the anti-canonical divisor decomposition



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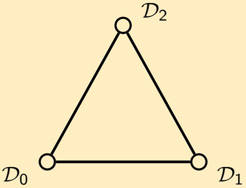

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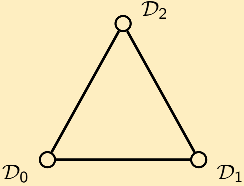

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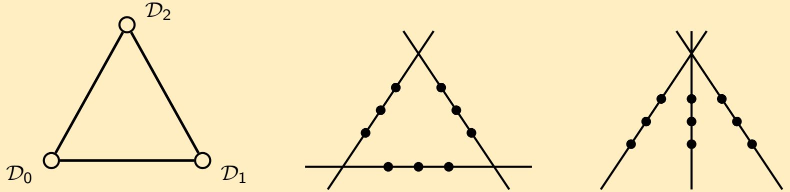

Dynkin diagram $A_2^{(1)}$ — $A_2^{(1)}$ surface (multiplicative) — $A_2^{(1)*}$ surface (additive)

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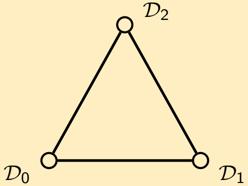

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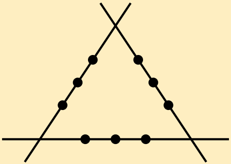

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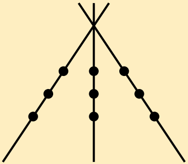

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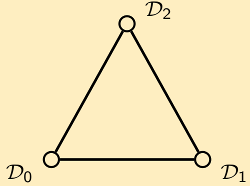

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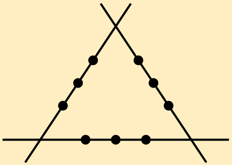

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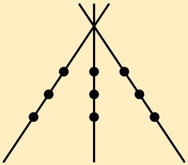

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Again, without the loss of generality (i.e., acting by affine transformations on each of the two $\mathbb{P}^1$ factors) we can assume that the component $D_1 = H_f - E_5 - E_6$ under the blowing down map projects to the line $f = \infty$ (and so there are two blowup points $p_5(\infty, b_5)$ and $p_6(\infty, b_6)$ on that line), the component $D_2 = H_g - E_7 - E_8$ projects to the line $g = \infty$ with points $p_7(-b_6, \infty)$ and $p_8(-b_8, \infty)$, and the component $D_0 = H_f + H_g - E_1 - E_2 - E_3 - E_4$ projects to the line $f + g = 0$.

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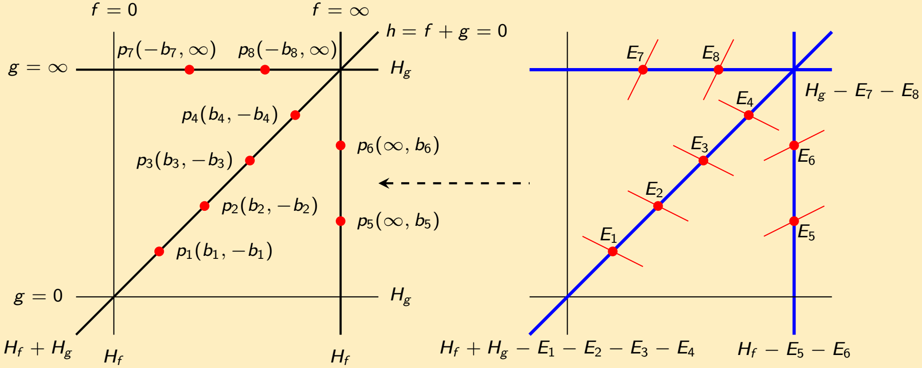

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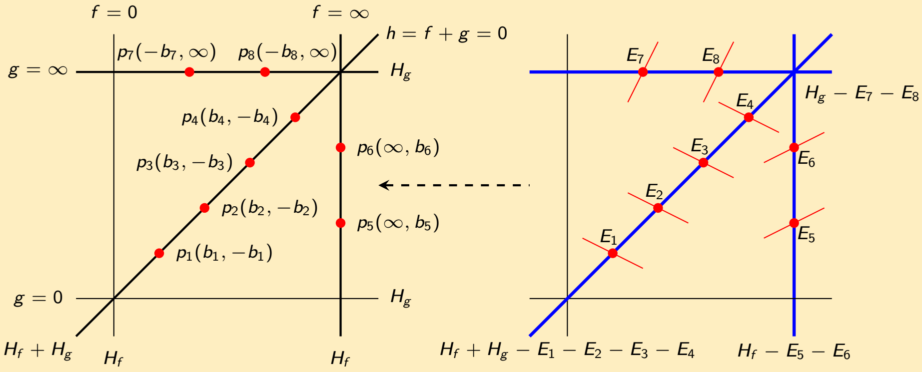

Note that the lines in the above configuration form a pole divisor of the symplectic form

$$\omega = \frac{df \wedge dg}{(f+g)} = -\frac{dF \wedge dg}{F(1+Fg)} = -\frac{df \wedge dG}{G(fG+1)} = \frac{dF \wedge dG}{(F+G)} = \frac{dh \wedge dg}{h} = \cdots$$

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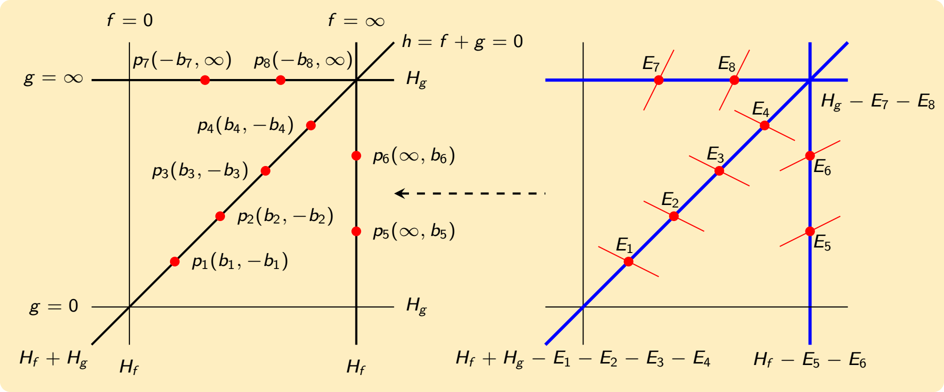

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However, there is still a two-parameter family of transformations preserving this configuration:

$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; f, g \Bigg) \sim \begin{pmatrix} \alpha b_1+\beta & \alpha b_2+\beta & \alpha b_3+\beta & \alpha b_4+\beta \\ \alpha b_5-\beta & \alpha b_6-\beta & \alpha b_7-\beta & \alpha b_8-\beta \end{pmatrix}; \alpha f+\beta, \alpha g-\beta \Bigg), \ \alpha \neq 0.$$

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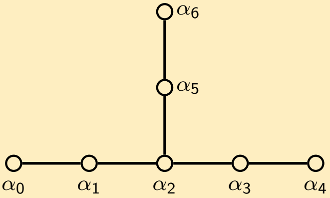
Symmetry sublattice $Q \triangleleft \mathrm{Pic}(\mathcal{X})$

$$Q = (\mathrm{Span}_{\mathbb{Z}}\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2\})^{\perp} = Q\left((A_2^{(1)})^{\perp}\right) = \mathrm{Span}_{\mathbb{Z}}\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\} = Q\left(E_6^{(1)}\right),$$

where the simple roots α_i are given by



$$\begin{aligned}
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Note also that $\delta = -\mathcal{K}_{\mathcal{X}} = \alpha_0 + 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6$.

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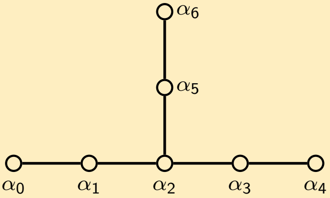

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The period mapping is the map

$$\chi : Q \to \mathbb{C}, \qquad \chi(\alpha_i) = a_i$$

defined on the simple roots and then extended by the linearity.

The Period Map

$$\chi(\alpha_i) = \chi\left([C_i^1] - [C_1^0]\right) = \int_{P_i}^{Q_i} \frac{1}{2\pi \mathrm{i}} \oint_{D_k} \omega$$
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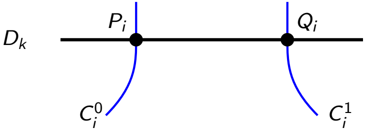

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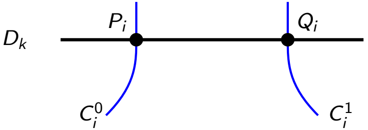
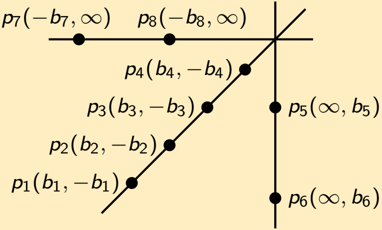


Examples of the Period Map computations



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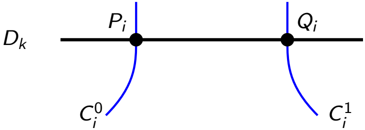

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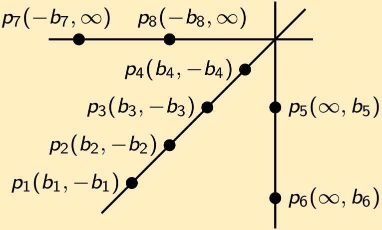

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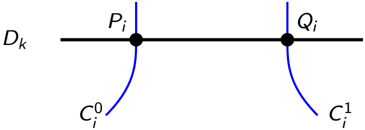

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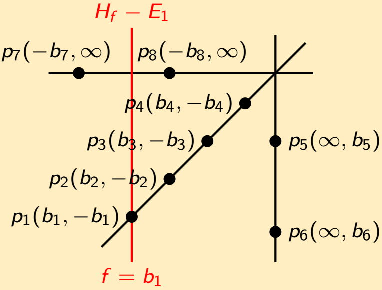

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The Period Map, $a_i = \chi(\alpha_i)$ are the *root variables*

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Parameterization by the root variables a_i

$$
\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; f, g \Bigg) = \left(\begin{matrix} b_1 & b_1 + a_2 & b_1 + a_1 + a_2 & b_1 + a_0 + a_1 + a_2 \\ a_5 - b_1 & a_5 + a_6 - b_1 & a_3 - b_1 & a_3 + a_4 - b_1 \end{matrix}; f, g \right),
$$

and so we see that b_1 is one free parameter (translation of the origin). To fix the global scaling parameter we usually normalize

$$
\begin{aligned}
\chi(\delta) = \chi(-\mathcal{K}_{\mathcal{X}}) &= \chi(a_0 + 2a_1 + 3a_2 + 2a_3 + a_4 + 2a_5 + a_6)\\
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$$

The usual normalization is to put $\chi(\delta) = 1$, and one can also ask the same for b_1. We will not do that, but we will require that, when resolving the normalization ambiguity, both $\chi(\delta)$ and b_1 are fixed — this ensures the group structure on the level of *elementary birational maps*.

The Extended Affine Weyl Symmetry Group $\widetilde{W}\left(E_6^{(1)}\right)$

The next step in understanding the structure of difference Painlevé equations of type d-$P\left(A_2^{(1)*}\right)$ is to describe the realization of the symmetry group in terms of elementary bilinear maps.

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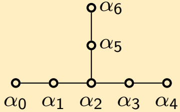
$$W(E_6^{(1)}) = \left\langle w_0, \dots, w_6 \;\middle|\; \begin{array}{cc} w_i^2 = e & \\ w_i \circ w_j = w_j \circ w_i & \text{when } \alpha_i \quad \alpha_j \\ w_i \circ w_j \circ w_i = w_j \circ w_i \circ w_j & \text{when } \alpha_i - \alpha_j \end{array} \right\rangle$$



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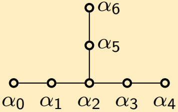
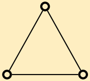


- The finite group of Dynkin diagram automorphisms

$$\mathrm{Aut}\left(E_6^{(1)}\right) \simeq \mathrm{Aut}\left(A_2^{(1)}\right) \simeq \mathbb{D}_3,$$



where $\mathbb{D}_3 = \{e, m_0, m_1, m_2, r, r^2\} = \langle m_0, r \mid m_0^2 = r^3 = e, m_0 r = r^2 m_0\rangle$ is the usual *dihedral group* of the symmetries of a triangle.

Theorem

Reflections w_i are induced by the following elementary birational mappings (also denoted by w_i) on the family $\mathcal{X}_{\mathbf{b}}$ fixing b_1 and $\chi(\delta)$ (we put $b_{i\dots k} = b_i + \cdots + b_k$, e.g., $b_{12} = b_1 + b_2$ and so on)

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Theorem

Reflections w_i are induced by the following elementary birational mappings (also denoted by w_i) on the family $\mathcal{X}_{\mathbf{b}}$ fixing b_1 and $\chi(\delta)$ (we put $b_{i\cdots k} = b_i + \cdots + b_k$, e.g., $b_{12} = b_1 + b_2$ and so on)

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$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; \begin{matrix} f \\ g \end{matrix} \overset{w_2}{\longmapsto} \begin{pmatrix} b_1 & b_{11} - b_2 & b_{13} - b_2 & b_{14} - b_2 \\ b_{52} - b_1 & b_{62} - b_1 & b_{72} - b_1 & b_{82} - b_1 \end{pmatrix}; \begin{matrix} f + b_1 - b_2 \\ g - b_1 + b_2 \end{matrix},$$

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$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; \begin{matrix} f \\ g \end{matrix} \overset{w_5}{\longmapsto} \begin{pmatrix} b_1 & b_{215} & b_{315} & b_{415} \\ -b_{115} & b_6 - b_{15} & b_7 & b_8 \end{pmatrix}; \begin{matrix} \frac{(f-b_1)(g-b_5)}{g+b_1} + b_1 \\ g - b_{15} \end{matrix},$$

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Sketch of the proof

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$\alpha_2 = \mathcal{E}_1 - \mathcal{E}_2$, $w_2 : \mathcal{E}_1 \leftrightarrow \mathcal{E}_2$, which swaps b_1 and b_2, but then we need to use the normalization freedom to ensure that b_1 is fixed,

$$\begin{pmatrix} b_2 & b_1 & b_4 & b_3 & ; & f \\ b_5 & b_6 & b_7 & b_8 & ; & g \end{pmatrix} \sim \begin{pmatrix} b_1 & b_{11} - b_2 & b_{13} - b_2 & b_{14} - b_2 & ; & f + b_1 - b_2 \\ b_{52} - b_1 & b_{62} - b_1 & b_{72} - b_1 & b_{82} - b_1 & ; & g - b_1 + b_2 \end{pmatrix}.$$

Since $\alpha_0 = \mathcal{E}_2 - \mathcal{E}_3$, $w_0 : \mathcal{E}_2 \leftrightarrow \mathcal{E}_3$, which just swaps the parameters b_2 and b_3. Similarly, since $\alpha_2 = \mathcal{E}_1 - \mathcal{E}_2$, $w_2 : \mathcal{E}_1 \leftrightarrow \mathcal{E}_2$, which swaps b_1 and b_2, but then we need to use the normalization freedom to ensure that b_1 is fixed,

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Consider now $\alpha_3 = \mathcal{H}_f - \mathcal{E}_1 - \mathcal{E}_7$. Then

$$w_3(\mathcal{H}_f) = \mathcal{H}_f, \quad w_3(\mathcal{H}_g) = \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_7, \quad w_3(\mathcal{E}_1) = \mathcal{H}_f - \mathcal{E}_7, \quad w_3(\mathcal{E}_7) = \mathcal{H}_f - \mathcal{E}_1,$$

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and $w_3(\mathcal{E}_i) = \mathcal{E}_i$ otherwise.
Thus, $w_3^{-1}(\mathcal{H}_{\bar{g}}) = \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_7$, i.e., $\bar{g}$ is a coordinate on a pencil of $(1,1)$ curves passing through $p_1(b_1, -b_1)$ and $p_7(-b_7, \infty)$:

$$\begin{aligned} |H_{\bar{g}}| &= \{Afg + Bf + Cg + D = 0 \mid -Ab_1^2 + (B - C)b_1 + D = -Ab_7 + C = 0\} \\ &= \{A(fg + b_7 g + b_1^2 + b_1 b_7) + b(f - b_1 = 0\} \implies \\ \bar{g} &= \frac{P(fg + b_7 g + b_1^2 + b_1 b_7) + Q(f - b_1)}{R(fg + b_7 g + b_1^2 + b_1 b_7) + S(f - b_1)}, \qquad \bar{f} = \frac{Lf + M}{Nf + T}. \end{aligned}$$

Since $\alpha_0 = \mathcal{E}_2 - \mathcal{E}_3$, $w_0 : \mathcal{E}_2 \leftrightarrow \mathcal{E}_3$, which just swaps the parameters b_2 and b_3. Similarly, since

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$$\begin{pmatrix} b_2 & b_1 & b_4 & b_3 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix} ; \begin{matrix} f \\ g \end{matrix} \sim \begin{pmatrix} b_1 & b_{11} - b_2 & b_{13} - b_2 & b_{14} - b_2 \\ b_{52} - b_1 & b_{62} - b_1 & b_{72} - b_1 & b_{82} - b_1 \end{pmatrix} ; \begin{matrix} f + b_1 - b_2 \\ g - b_1 + b_2 \end{matrix}.$$

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Since $\bar{g}(-b_8, \infty) = \infty$, $R = 0$, and since $\bar{f}(\infty, b_5) = \infty$, $N = 0$.

Sketch of the proof (cont.)

So we have

$$\bar{f} = Lf + M, \qquad \bar{g} = P\frac{fg + b_7 g + b_1^2 + b_1 b_7}{f - b_1} + Q.$$

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Then we have

$$\begin{aligned}
(\bar{b}_1, -\bar{b}_1) &= (\bar{f}, \bar{g})(f = -b_7) = (-Lb_7 + M, -Pb_1 + Q),\\
(\bar{b}_2, -\bar{b}_2) &= (\bar{f}, \bar{g})(b_2, -b_2) = (Lb_2 + M, -P(b_1 + b_2 + b_7) + Q),\\
(\bar{b}_3, -\bar{b}_3) &= (\bar{f}, \bar{g})(b_3, -b_3) = (Lb_3 + M, -P(b_1 + b_3 + b_7) + Q),\\
(\bar{b}_4, -\bar{b}_4) &= (\bar{f}, \bar{g})(b_4, -b_4) = (Lb_4 + M, -P(b_1 + b_4 + b_7) + Q),\\
(\infty, \bar{b}_5) &= (\bar{f}, \bar{g})(\infty, b_5) = (\infty, Pb_5 + Q),\\
(\infty, \bar{b}_6) &= (\bar{f}, \bar{g})(\infty, b_6) = (\infty, Pb_6 + Q),\\
(-\bar{b}_7, \infty) &= (\bar{f}, \bar{g})(f = b_1) = (Lb_1 + M, \infty),\\
(-\bar{b}_8, \infty) &= (\bar{f}, \bar{g})(-b_8, \infty) = (-Lb_8 + M, \infty).
\end{aligned}$$

Sketch of the proof (cont.)

So we have

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(\bar{b}_3, -\bar{b}_3) &= (\bar{f}, \bar{g})(b_3, -b_3) = (Lb_3 + M, -P(b_1 + b_3 + b_7) + Q),\\
(\bar{b}_4, -\bar{b}_4) &= (\bar{f}, \bar{g})(b_4, -b_4) = (Lb_4 + M, -P(b_1 + b_4 + b_7) + Q),\\
(\infty, \bar{b}_5) &= (\bar{f}, \bar{g})(\infty, b_5) = (\infty, Pb_5 + Q),\\
(\infty, \bar{b}_6) &= (\bar{f}, \bar{g})(\infty, b_6) = (\infty, Pb_6 + Q),\\
(-\bar{b}_7, \infty) &= (\bar{f}, \bar{g})(f = b_1) = (Lb_1 + M, \infty),\\
(-\bar{b}_8, \infty) &= (\bar{f}, \bar{g})(-b_8, \infty) = (-Lb_8 + M, \infty).
\end{aligned}$$

The first four equations give $L = P$ and then to preserve $\chi(\delta)$ we must have $L = P = 1$. Then, to fix b_1, $M = b_1 + b_7$ and $Q = 0$, which gives the required map.

Other cases are similar to the ones considered.

The Automorphism Group $\mathrm{Aut}(A_2^{(1)}) \simeq \mathrm{Aut}(E_6^{(1)}) \simeq \mathbb{D}_3$

Theorem

The acton of the automorphisms on the Picard lattice $\mathrm{Pic}(\mathcal{X})$*, the symmetry sub-lattice* $\mathrm{Span}_{\mathbb{Z}}\{\alpha_i\}$ *and the surface sub-lattice* $\mathrm{Span}_{\mathbb{Z}}\{\mathcal{D}_i\}$ *is given by:*

$$m_0 = (\mathcal{D}_1\mathcal{D}_2) = (\alpha_3\alpha_5)(\alpha_4\alpha_6),$$

$$\mathcal{H}_f \to \mathcal{H}_g, \quad \mathcal{E}_1 \to \mathcal{E}_1, \quad \mathcal{E}_3 \to \mathcal{E}_3, \quad \mathcal{E}_5 \to \mathcal{E}_7, \quad \mathcal{E}_7 \to \mathcal{E}_5,$$

$$\mathcal{H}_g \to \mathcal{H}_f, \quad \mathcal{E}_2 \to \mathcal{E}_2, \quad \mathcal{E}_4 \to \mathcal{E}_4, \quad \mathcal{E}_6 \to \mathcal{E}_8, \quad \mathcal{E}_8 \to \mathcal{E}_6;$$

$$m_1 = (\mathcal{D}_0\mathcal{D}_2) = (\alpha_0\alpha_4)(\alpha_1\alpha_3),$$

$$\mathcal{H}_f \to \mathcal{H}_f, \quad \mathcal{E}_1 \to \mathcal{H}_f - \mathcal{E}_2, \quad \mathcal{E}_3 \to \mathcal{E}_7, \quad \mathcal{E}_5 \to \mathcal{E}_5, \quad \mathcal{E}_7 \to \mathcal{E}_3,$$

$$\mathcal{H}_g \to \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2, \quad \mathcal{E}_2 \to \mathcal{H}_f - \mathcal{E}_1, \quad \mathcal{E}_4 \to \mathcal{E}_8, \quad \mathcal{E}_6 \to \mathcal{E}_6, \quad \mathcal{E}_8 \to \mathcal{E}_4;$$

$$m_2 = (\mathcal{D}_0\mathcal{D}_1) = (\alpha_0\alpha_6)(\alpha_1\alpha_5),$$

$$\mathcal{H}_f \to \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2, \quad \mathcal{E}_1 \to \mathcal{H}_g - \mathcal{E}_2, \quad \mathcal{E}_3 \to \mathcal{E}_5, \quad \mathcal{E}_5 \to \mathcal{E}_3, \quad \mathcal{E}_7 \to \mathcal{E}_7,$$

$$\mathcal{H}_g \to \mathcal{H}_g, \quad \mathcal{E}_2 \to \mathcal{H}_g - \mathcal{E}_1, \quad \mathcal{E}_4 \to \mathcal{E}_6, \quad \mathcal{E}_6 \to \mathcal{E}_4, \quad \mathcal{E}_8 \to \mathcal{E}_8;$$

$$r = (\mathcal{D}_0\mathcal{D}_1\mathcal{D}_2) = (\alpha_0\alpha_6\alpha_4)(\alpha_1\alpha_5\alpha_3),$$

$$\mathcal{H}_f \to \mathcal{H}_g, \quad \mathcal{E}_1 \to \mathcal{H}_g - \mathcal{E}_2, \quad \mathcal{E}_3 \to \mathcal{E}_5, \quad \mathcal{E}_5 \to \mathcal{E}_7, \quad \mathcal{E}_7 \to \mathcal{E}_3,$$

$$\mathcal{H}_g \to \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2, \quad \mathcal{E}_2 \to \mathcal{H}_g - \mathcal{E}_1, \quad \mathcal{E}_4 \to \mathcal{E}_6, \quad \mathcal{E}_6 \to \mathcal{E}_8, \quad \mathcal{E}_8 \to \mathcal{E}_4;$$

$$r^2 = (\mathcal{D}_0\mathcal{D}_2\mathcal{D}_1) = (\alpha_0\alpha_4\alpha_6)(\alpha_1\alpha_3\alpha_5),$$

$$\mathcal{H}_f \to \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2, \quad \mathcal{E}_1 \to \mathcal{H}_f - \mathcal{E}_2, \quad \mathcal{E}_3 \to \mathcal{E}_7, \quad \mathcal{E}_5 \to \mathcal{E}_3, \quad \mathcal{E}_7 \to \mathcal{E}_5,$$

$$\mathcal{H}_g \to \mathcal{H}_f, \quad \mathcal{E}_2 \to \mathcal{H}_f - \mathcal{E}_1, \quad \mathcal{E}_4 \to \mathcal{E}_8, \quad \mathcal{E}_6 \to \mathcal{E}_4, \quad \mathcal{E}_8 \to \mathcal{E}_6.$$

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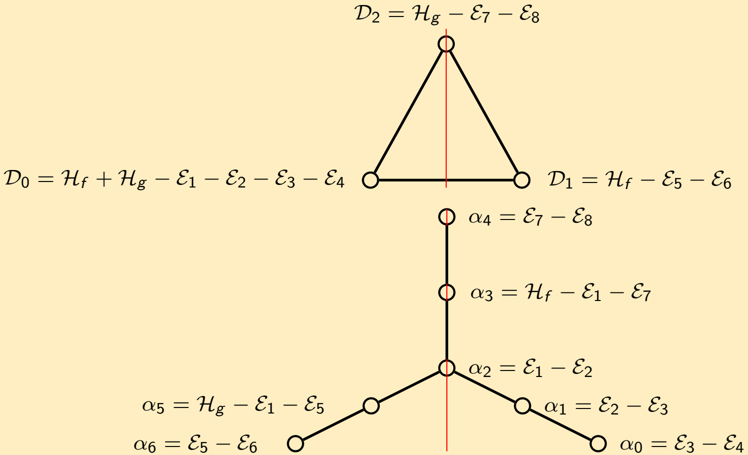

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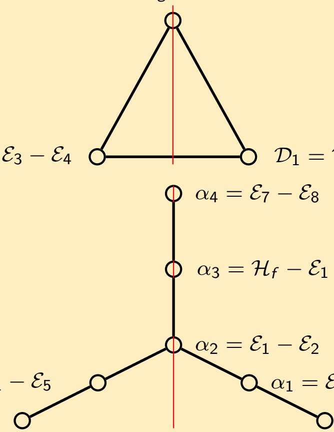

Hence, m_2 is given by

$$\begin{array}{lllll} \mathcal{H}_f \to \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2, & \mathcal{E}_1 \to \mathcal{H}_g - \mathcal{E}_2, & \mathcal{E}_3 \to \mathcal{E}_5, & \mathcal{E}_5 \to \mathcal{E}_3, & \mathcal{E}_7 \to \mathcal{E}_7, \\ \mathcal{H}_g \to \mathcal{H}_g, & \mathcal{E}_2 \to \mathcal{H}_g - \mathcal{E}_1, & \mathcal{E}_4 \to \mathcal{E}_6, & \mathcal{E}_6 \to \mathcal{E}_4, & \mathcal{E}_8 \to \mathcal{E}_8; \end{array}$$

The Automorphism Group $\mathrm{Aut}(A_2^{(1)}) \simeq \mathrm{Aut}(E_6^{(1)}) \simeq \mathbb{D}_3$

Theorem

The automorphisms are given by the following elementary birational maps on the family $\mathcal{X}_{\mathbf{b}}$ fixing b_1 and $\chi(\delta)$

$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; \begin{matrix} f \\ g \end{matrix} \overset{m_0}{\longmapsto} \begin{pmatrix} b_1 & b_2 & b_4 & b_3 \\ b_7 & b_8 & b_5 & b_6 \end{pmatrix}; \begin{matrix} -f \\ -g \end{matrix},$$

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$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; \begin{matrix} f \\ g \end{matrix} \overset{m_2}{\longmapsto} \begin{pmatrix} b_1 & b_2 & b_{125} & b_{126} \\ b_3 - b_{12} & b_4 - b_{12} & b_7 & b_8 \end{pmatrix}; \begin{matrix} \frac{f(g + b_{12}) - b_1 b_2}{f+g} \\ -g - b_{12} \end{matrix},$$

$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; \begin{matrix} f \\ g \end{matrix} \overset{r}{\longmapsto} \begin{pmatrix} b_1 & b_2 & b_{127} & b_{128} \\ b_3 - b_{12} & b_4 - b_{12} & b_5 & b_6 \end{pmatrix}; \begin{matrix} -\frac{g(f - b_{12}) - b_1 b_2}{f+g} \\ f - b_{12} \end{matrix},$$

$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; \begin{matrix} f \\ g \end{matrix} \overset{r^2}{\longmapsto} \begin{pmatrix} b_1 & b_2 & b_{125} & b_{126} \\ b_7 & b_8 & b_3 - b_{12} & b_4 - b_{12} \end{pmatrix}; \begin{matrix} g + b_{12} \\ -\frac{f(g + b_{12}) - b_1 b_2}{f+g} \end{matrix}.$$

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Proof is similar to the previous theorem. Notice that the group structure is preserved on the level of the maps.

The Semi-Direct Product Structure

The extended affine Weyl group $\widetilde{W}(E_6^{(1)})$ is a semi-direct product of its normal subgroup $W(E_6^{(1)}) \triangleleft \widetilde{W}(E_6^{(1)})$ and the subgroup of the diagram automorphisms $\mathrm{Aut}(E_6^{(1)})$,

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We have just described the group structure of $W(E_6^{(1)})$ and $\mathrm{Aut}(E_6^{(1)})$ using generators and relations, so it remains to give the action of $\mathrm{Aut}(E_6^{(1)})$ on $W(E_6^{(1)})$.

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But elements of $\mathrm{Aut}(E_6^{(1)})$ act as permutations of the simple roots α_i, and so the action is just the corresponding permutation of the corresponding reflections, $\sigma_t w_{\alpha_i} \sigma_t^{-1} = w_{t(\alpha_i)}$, where t is the permutation of α_i's corresponding to σ_t.

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Example: $\sigma_1 = \sigma_{m_1} = (\alpha_0\alpha_4)(\alpha_1\alpha_3)$ acts as

$$\sigma_1 w_0 \sigma_1 = w_4, \quad \sigma_1 w_4 \sigma_1 = w_0, \quad \sigma_1 w_1 \sigma_1 = w_3, \quad \sigma_1 w_3 \sigma_1 = w_1, \qquad \sigma_1 w_i \sigma_1 = w_i \quad \text{otherwise}.$$

Decomposition of Translation Elements

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If $w(\alpha_i) < 0$, then

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As an example, consider the following translational mapping:

$$\varphi_* : (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \mapsto (\alpha_0, \alpha_1, \alpha_2, \alpha_3 + \delta, \alpha_4, \alpha_5 - \delta, \alpha_6),$$

where $\delta = \alpha_0 + 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6$ as usual.

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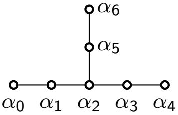
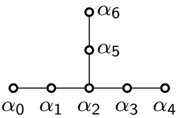
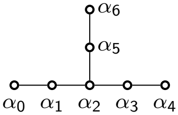
Put
$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6).$$

Then the algorithm works as follows:


α_6
α_5
α_0 α_1 α_2 α_3 α_4

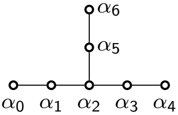


$$\varphi_*(\alpha) = (\alpha_0, \alpha_1, \alpha_2, \alpha_3 + \delta, \alpha_4, \alpha_5 - \delta, \alpha_6),$$



$$\varphi_*(\alpha) = (\alpha_0, \alpha_1, \alpha_2, \alpha_3 + \delta, \alpha_4, \alpha_5 - \delta, \alpha_6),$$

$$\left(\varphi_*^{(1)} = \varphi_* \circ w_5\right)(\alpha) = (\alpha_0, \alpha_1, \alpha_{25} - \delta, \alpha_3 + \delta, \alpha_4, \delta - \alpha_5, \alpha_{56} - \delta),$$



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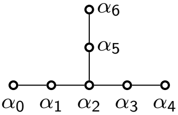

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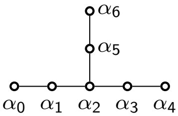

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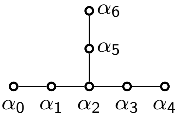
$$\left(\varphi_*^{(4)} = \varphi_*^{(3)} \circ w_1\right)(\alpha) = (\alpha_{0125} - \delta, \delta - \alpha_{125}, \alpha_1, \alpha_{235}, \alpha_4, \alpha_{256} - \delta, \delta - \alpha_{56}),$$



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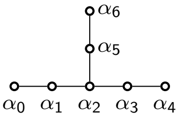

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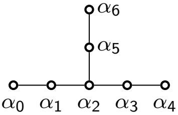
$$\left(\varphi_*^{(6)} = \varphi_*^{(5)} \circ w_5\right)(\alpha) = (\delta - \alpha_{0125}, \alpha_0, \alpha_{1256} - \delta, \alpha_{235}, \alpha_4, \delta - \alpha_{256}, \alpha_2),$$



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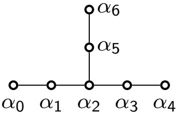
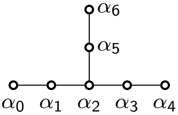
$$\left(\varphi_*^{(7)} = \varphi_*^{(6)} \circ w_2\right)(\alpha) = (\alpha_{12233456}, -\alpha_{1223345}, \alpha_{01223345}, -\alpha_{01234}, \alpha_4, \alpha_1, \alpha_2),$$



$$
\begin{aligned}
\varphi_*(\alpha) &= (\alpha_0, \alpha_1, \alpha_2, \alpha_3 + \delta, \alpha_4, \alpha_5 - \delta, \alpha_6),\\
\left(\varphi_*^{(1)} = \varphi_* \circ w_5\right)(\alpha) &= (\alpha_0, \alpha_1, \alpha_{25} - \delta, \alpha_3 + \delta, \alpha_4, \delta - \alpha_5, \alpha_{56} - \delta),\\
\left(\varphi_*^{(2)} = \varphi_*^{(1)} \circ w_6\right)(\alpha) &= (\alpha_0, \alpha_1, \alpha_{25} - \delta, \alpha_3 + \delta, \alpha_4, \alpha_6, \delta - \alpha_{56}),\\
\left(\varphi_*^{(3)} = \varphi_*^{(2)} \circ w_2\right)(\alpha) &= (\alpha_0, \alpha_{125} - \delta, \delta - \alpha_{25}, \alpha_{235}, \alpha_4, \alpha_{256} - \delta, \delta - \alpha_{56}),\\
\left(\varphi_*^{(4)} = \varphi_*^{(3)} \circ w_1\right)(\alpha) &= (\alpha_{0125} - \delta, \delta - \alpha_{125}, \alpha_1, \alpha_{235}, \alpha_4, \alpha_{256} - \delta, \delta - \alpha_{56}),\\
\left(\varphi_*^{(5)} = \varphi_*^{(4)} \circ w_0\right)(\alpha) &= (\delta - \alpha_{0125}, \alpha_0, \alpha_1, \alpha_{235}, \alpha_4, \alpha_{256} - \delta, \delta - \alpha_{56}),\\
\left(\varphi_*^{(6)} = \varphi_*^{(5)} \circ w_5\right)(\alpha) &= (\delta - \alpha_{0125}, \alpha_0, \alpha_{1256} - \delta, \alpha_{235}, \alpha_4, \delta - \alpha_{256}, \alpha_2),\\
\left(\varphi_*^{(7)} = \varphi_*^{(6)} \circ w_2\right)(\alpha) &= (\alpha_{12233456}, -\alpha_{1223345}, \alpha_{01223345}, -\alpha_{01234}, \alpha_4, \alpha_1, \alpha_2),\\
\left(\varphi_*^{(8)} = \varphi_*^{(7)} \circ w_1\right)(\alpha) &= (\alpha_6, \alpha_{1223345}, \alpha_0, -\alpha_{01234}, \alpha_4, \alpha_1, \alpha_2),
\end{aligned}
$$



$$\varphi_*(\alpha) = (\alpha_0, \alpha_1, \alpha_2, \alpha_3 + \delta, \alpha_4, \alpha_5 - \delta, \alpha_6),$$

$$\left(\varphi_*^{(1)} = \varphi_* \circ w_5\right)(\alpha) = (\alpha_0, \alpha_1, \alpha_{25} - \delta, \alpha_3 + \delta, \alpha_4, \delta - \alpha_5, \alpha_{56} - \delta),$$

$$\left(\varphi_*^{(2)} = \varphi_*^{(1)} \circ w_6\right)(\alpha) = (\alpha_0, \alpha_1, \alpha_{25} - \delta, \alpha_3 + \delta, \alpha_4, \alpha_6, \delta - \alpha_{56}),$$

$$\left(\varphi_*^{(3)} = \varphi_*^{(2)} \circ w_2\right)(\alpha) = (\alpha_0, \alpha_{125} - \delta, \delta - \alpha_{25}, \alpha_{235}, \alpha_4, \alpha_{256} - \delta, \delta - \alpha_{56}),$$

$$\left(\varphi_*^{(4)} = \varphi_*^{(3)} \circ w_1\right)(\alpha) = (\alpha_{0125} - \delta, \delta - \alpha_{125}, \alpha_1, \alpha_{235}, \alpha_4, \alpha_{256} - \delta, \delta - \alpha_{56}),$$

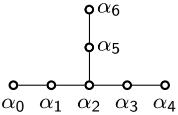
$$\left(\varphi_*^{(5)} = \varphi_*^{(4)} \circ w_0\right)(\alpha) = (\delta - \alpha_{0125}, \alpha_0, \alpha_1, \alpha_{235}, \alpha_4, \alpha_{256} - \delta, \delta - \alpha_{56}),$$

$$\left(\varphi_*^{(6)} = \varphi_*^{(5)} \circ w_5\right)(\alpha) = (\delta - \alpha_{0125}, \alpha_0, \alpha_{1256} - \delta, \alpha_{235}, \alpha_4, \delta - \alpha_{256}, \alpha_2),$$

$$\left(\varphi_*^{(7)} = \varphi_*^{(6)} \circ w_2\right)(\alpha) = (\alpha_{12233456}, -\alpha_{1223345}, \alpha_{01223345}, -\alpha_{01234}, \alpha_4, \alpha_1, \alpha_2),$$

$$\left(\varphi_*^{(8)} = \varphi_*^{(7)} \circ w_1\right)(\alpha) = (\alpha_6, \alpha_{1223345}, \alpha_0, -\alpha_{01234}, \alpha_4, \alpha_1, \alpha_2),$$

$$\left(\varphi_*^{(9)} = \varphi_*^{(8)} \circ w_3\right)(\alpha) = (\alpha_6, \alpha_{1223345}, -\alpha_{1234}, \alpha_{01234}, -\alpha_{0123}, \alpha_1, \alpha_2),$$



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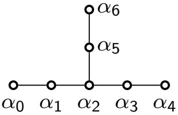

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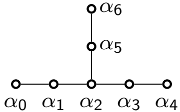

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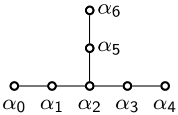

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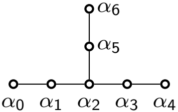

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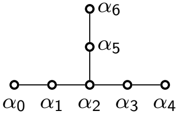

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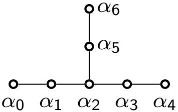

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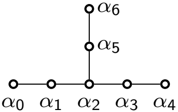

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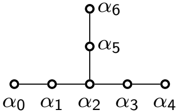

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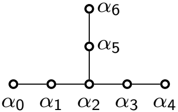

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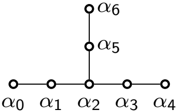

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Thus,

$$\varphi_* = \sigma_r \circ w_5 \circ w_2 \circ w_3 \circ w_6 \circ w_5 \circ w_2 \circ w_4 \circ w_3 \circ w_1 \circ w_2 \circ w_5 \circ w_0 \circ w_1 \circ w_2 \circ w_6 \circ w_5$$

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First let us review these equations.

The following example of a d-$P(A_2^{(1)*})$ equation was first obtained by B. Grammaticos, A. Ramani, and Y. Ohta back around 1996 by applying the singularity confinement criterion to deautonomization of an integrable discrete autonomous mapping; due to the simplicity structure of the equation we will refer to it as a *model example*.

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The Model Example of d-$P(A_2^{(1)*})$

We consider a birational map $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ with parameters $b_1, \ldots, b_8$:

$$\varphi : \quad \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; f, g \Biggr) \mapsto \Biggl(\begin{matrix} \bar{b}_1 & \bar{b}_2 & \bar{b}_3 & \bar{b}_4 \\ \bar{b}_5 & \bar{b}_6 & \bar{b}_7 & \bar{b}_8 \end{matrix}; \bar{f}, \bar{g} \Biggr),$$

$$\delta = b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8$$

$$\bar{b}_1 = b_1, \quad \bar{b}_3 = b_3, \quad \bar{b}_5 = b_5 + \delta, \quad \bar{b}_7 = b_7 - \delta$$

$$\bar{b}_2 = b_2, \quad \bar{b}_4 = b_4, \quad \bar{b}_6 = b_6 + \delta, \quad \bar{b}_8 = b_8 - \delta,$$

and $\bar{f}$ and $\bar{g}$ are given by the equation

$$\begin{cases} (f+g)(\bar{f}+g) = \dfrac{(g+b_1)(g+b_2)(g+b_3)(g+b_4)}{(g-b_5)(g-b_6)} \\ (\bar{f}+g)(\bar{f}+\bar{g}) = \dfrac{(\bar{f}-\bar{b}_1)(\bar{f}-\bar{b}_2)(\bar{f}-\bar{b}_3)(\bar{f}-\bar{b}_4)}{(\bar{f}+\bar{b}_7)(\bar{f}+\bar{b}_8)} \end{cases}.$$

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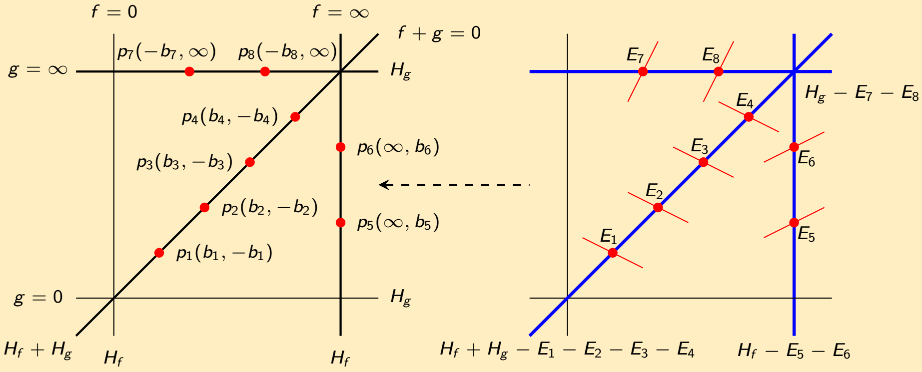

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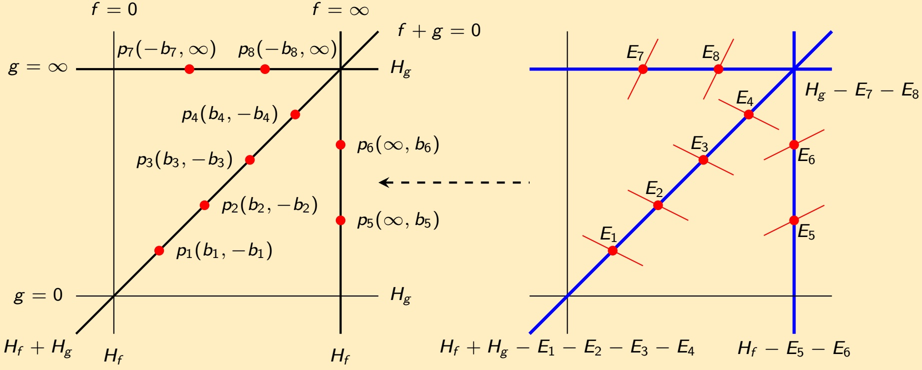

Now let us compute the action of this mapping on $\mathrm{Pic}(\mathcal{X})$

The action of φ_* on Pic($\mathcal{X}$)

Finally, we compute the action of φ_* on Pic($\mathcal{X}$) to be

$$
\begin{aligned}
\mathcal{H}_f &\mapsto 6\mathcal{H}_f + 3\mathcal{H}_g - 2\mathcal{E}_1 - 2\mathcal{E}_2 - 2\mathcal{E}_3 - 2\mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6 - 3\mathcal{E}_7 - 3\mathcal{E}_8,\\
\mathcal{H}_g &\mapsto 3\mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_1 &\mapsto 2\mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_2 &\mapsto 2\mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_3 &\mapsto 2\mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_4 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_4 &\mapsto 2\mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_5 &\mapsto 3\mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8,\\
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\end{aligned}
$$

The action of φ_* on Pic($\mathcal{X}$)

Finally, we compute the action of φ_* on Pic($\mathcal{X}$) to be

$$\begin{aligned}
\mathcal{H}_f &\mapsto 6\mathcal{H}_f + 3\mathcal{H}_g - 2\mathcal{E}_1 - 2\mathcal{E}_2 - 2\mathcal{E}_3 - 2\mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6 - 3\mathcal{E}_7 - 3\mathcal{E}_8,\\
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\mathcal{E}_2 &\mapsto 2\mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_7 - \mathcal{E}_8,\\
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\end{aligned}$$

and so the induced action φ_* on the sub-lattice $R^\perp$ is given by the following *translation*:

$$(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6) \mapsto (\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6) + (0,0,0,1,0,-1,0)\delta,$$

as well as the permutation $\sigma_r = (\mathcal{D}_0\mathcal{D}_1\mathcal{D}_2)$ of the irreducible components of $-K_{\mathcal{X}}$.

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as well as the permutation $\sigma_r = (\mathcal{D}_0\mathcal{D}_1\mathcal{D}_2)$ of the irreducible components of $-K_{\mathcal{X}}$.

Hence $\varphi_* = \sigma_r \circ w_5 \circ w_2 \circ w_3 \circ w_6 \circ w_5 \circ w_2 \circ w_4 \circ w_3 \circ w_1 \circ w_2 \circ w_5 \circ w_0 \circ w_1 \circ w_2 \circ w_6 \circ w_5$.

Sakai's Classification Scheme for Discrete Painlevé Equations.

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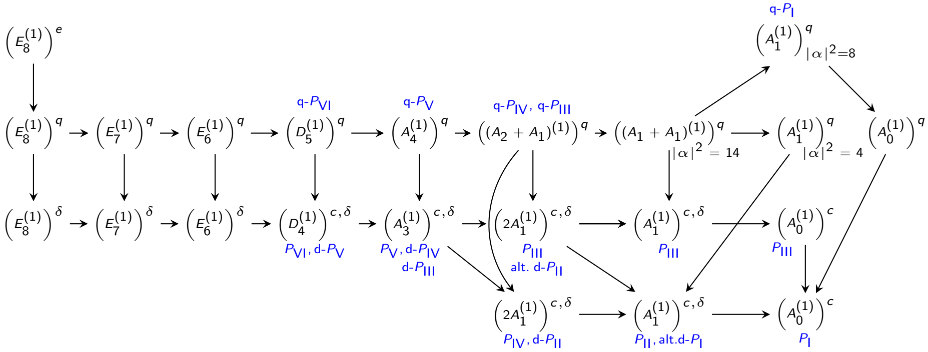

Symmetry-type classification scheme for Painlevé equations

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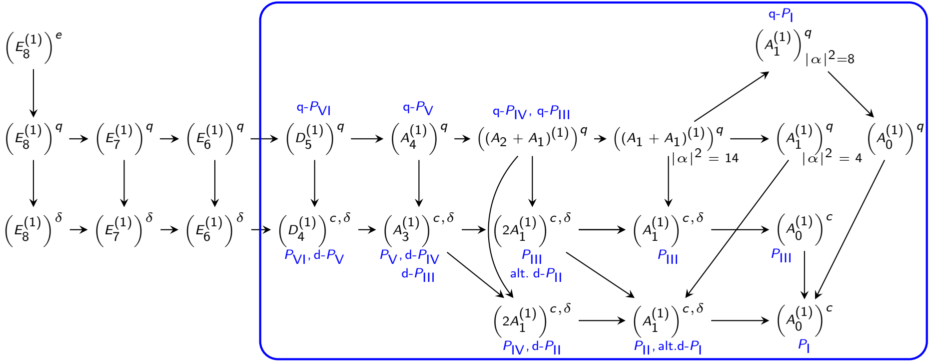

The differential part of the classification scheme

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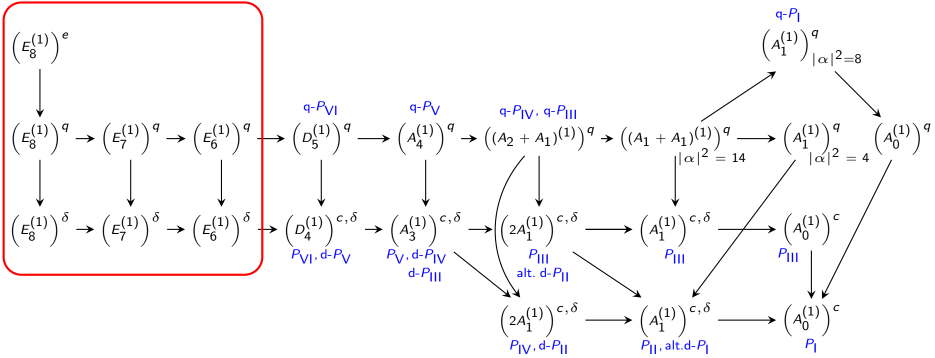

The purely discrete part of the classification scheme: why Painlevé?

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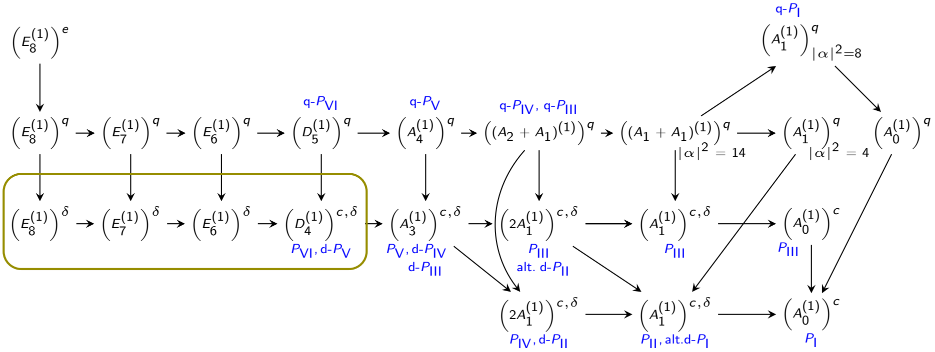

Isomonodromic approach: difference Painlevé equations as reductions from Schlesinger transformations of Fuchsian systems

Difference Painlevé Equation of Type d-$P(A_2^{(1)*})$: Schlesinger Transformations

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$$\left(E_8^{(1)}\right)^\delta \to \left(E_7^{(1)}\right)^\delta \to \left(E_6^{(1)}\right)^\delta \to \cdots \text{(symmetry) or } \left(A_0^{(1)**}\right)^\delta \to \left(A_1^{(1)*}\right)^\delta \to \left(A_2^{(1)*}\right)^\delta \to \cdots \text{(surface)}$$

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P.Boalch has identified the Fuchsian systems whose Schlesinger transformations have the required symmetry type (spectral type $1^3 1^3 1^3$ for d-$P(\widetilde{A}_2^*)$).

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Take $n = 2$ finite poles $z_0 = 0$, $z_1 = 1$, matrix size $m = 3$, and $\mathrm{rank}(\mathbf{A}_i) = 2$:

$$\mathbf{A}(z) = \frac{\mathbf{A}_0}{z} + \frac{\mathbf{A}_1}{z-1}, \qquad \mathbf{A}_i = \mathbf{B}_i \mathbf{C}_i^{\dagger} = \begin{bmatrix} \mathbf{b}_{i,1} & \mathbf{b}_{i,2} \end{bmatrix} \begin{bmatrix} \mathbf{c}_i^{1\dagger} \\ \mathbf{c}_i^{2\dagger} \end{bmatrix}.$$

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The corresponding *Riemann scheme* and the *Fuchs relation* are

$$\left\{ \begin{matrix} z=0 & z=1 & z=\infty \\ \theta_0^1 & \theta_1^1 & \kappa_1 \\ \theta_0^2 & \theta_1^2 & \kappa_2 \\ 0 & 0 & \kappa_3 \end{matrix} \right\}, \qquad \theta_0^1 + \theta_0^2 + \theta_1^1 + \theta_1^2 + \sum_{j=1}^3 \kappa_j = 0.$$

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No continuous deformations but non-trivial Schlesinger transformations.

Using various gauge transformations we can normalize the **b**-vectors, and then use the condition $\mathbf{C}_i^{\dagger}\mathbf{B}_i = \hat{}_i$ to parameterize the $\mathbf{c}^{\dagger}$-vectors:

$$\mathbf{B}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{C}_0^{\dagger} = \begin{bmatrix} \theta_0^1 & 0 & \alpha \\ 0 & \theta_0^2 & \beta \end{bmatrix}, \mathbf{B}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \mathbf{C}_1^{\dagger} = \begin{bmatrix} -\gamma - \theta_1^1 & \gamma & \theta_1^1 \\ \theta_1^2 - \delta & \delta & 0 \end{bmatrix}.$$

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Requiring that the eigenvalues of $\mathbf{A}_\infty = -\mathbf{A}_0 - \mathbf{A}_1$ are κ_1, κ_2, and κ_3:

$$\begin{aligned} \operatorname{tr}(\mathbf{A}_\infty) &= \kappa_1 + \kappa_2 + \kappa_3 \qquad \text{(the Fuchs relation)} \\ |\mathbf{A}_\infty|_{11} + |\mathbf{A}_\infty|_{22} + |\mathbf{A}_\infty|_{33} &= \kappa_2\kappa_3 + \kappa_3\kappa_1 + \kappa_1\kappa_2 \\ \det(\mathbf{A}_\infty) &= \kappa_1\kappa_2\kappa_3 \end{aligned}$$

imposes two linear constraints on four parameters α, β, γ, and δ.

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$$\begin{aligned} (\gamma + \delta + \theta_1^1 - \theta_1^2)\alpha - (\gamma + \delta)\beta &= \kappa_2\kappa_3 + \kappa_3\kappa_1 + \kappa_1\kappa_2 + (\theta_0^2 - \theta_0^1)\delta \\ &\quad - (\theta_0^2 + \theta_1^1)(\theta_0^1 + \theta_1^2) - \theta_0^2\theta_1^1), \\ -(\theta_0^2(\gamma + \delta + \theta_1^1 - \theta_1^2) + \theta_1^2\gamma + \theta_1^1\delta)\alpha + (\theta_0^1(\gamma + \delta) + \theta_1^2\gamma + \theta_1^1\delta)\beta &= \kappa_1\kappa_2\kappa_3 \\ &\quad + \theta_1^1((\theta_0^1 - \theta_0^2)\delta + \theta_0^2(\theta_0^1 + \theta_1^2)). \end{aligned}$$

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Notice that the coefficients of the matrix of the above linear system are written in terms of the expressions $\gamma + \delta$, $\gamma + \delta + \theta_1^1 - \theta_1^2$, and $\theta_1^2\gamma + \theta_1^1\delta$.

Choose parameterization variables x and y to simplify the structure of the substitution rule (matrix entries and *the determinant*):

$$x = \frac{(\gamma + \delta)(\theta_0^1 - \theta_0^2)}{\theta_1^1 - \theta_1^2}, \qquad y = \frac{\theta_1^2 \gamma + \theta_1^1 \delta}{\gamma + \delta + \theta_1^1 - \theta_1^2}.$$

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This gives:

$$\alpha(x,y) = \frac{\left(yr_1 + \frac{x(\theta_0^2 r_1 + r_2)}{x+\theta_0^1-\theta_0^2}\right)}{(x+y)(\theta_1^1-\theta_1^2)}, \qquad \beta(x,y) = \frac{((y+\theta_0^2)r_1 + r_2)}{(x+y)(\theta_1^1-\theta_1^2)},$$

where r_1 and r_2 are the right-hand-sides of our linear system on α and β

$$\begin{aligned} r_1 = r_1(x,y) &= \kappa_1\kappa_2 + \kappa_2\kappa_3 + \kappa_3\kappa_1 - (y-\theta_1^2)(x-\theta_0^2) - \theta_0^1(y+\theta_0^2) \\ &\qquad - \theta_1^1(\theta_0^1+\theta_0^2+\theta_1^2), \\ r_2 = r_2(x,y) &= \kappa_1\kappa_2\kappa_3 + \theta_1^1((y-\theta_1^2)(x-\theta_0^2) + \theta_0^1(y+\theta_0^2)). \end{aligned}$$

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Schlesinger evolution equations give us the map $\psi : (x,y) \to (\bar{x},\bar{y})$:

$$\begin{cases} \bar{x} = \dfrac{(\alpha-\beta)(\alpha x(\theta_1^1-\theta_1^2) + (1+\theta_0^2)(x(y-\theta_1^2) + y(\theta_0^1-\theta_0^2)))}{(\alpha-\beta)(x(y-\theta_1^2) + (\theta_0^1-\theta_0^2)y) - \alpha(\theta_1^1+1)(\theta_0^1-\theta_0^2)} \\ \bar{y} = \dfrac{(\alpha-\beta)(y(x+\theta_0^1-\theta_0^2) - \theta_1^2 x)}{\alpha(\theta_0^1-\theta_0^2)} \end{cases}.$$

Choose parameterization variables x and y to simplify the structure of the substitution rule (matrix entries and *the determinant*):

$$x = \frac{(\gamma+\delta)(\theta_0^1 - \theta_0^2)}{\theta_1^1 - \theta_1^2}, \qquad y = \frac{\theta_1^2\gamma + \theta_1^1\delta}{\gamma+\delta+\theta_1^1-\theta_1^2}.$$

This gives:

$$\alpha(x,y) = \frac{\left(yr_1 + \frac{x(\theta_0^2 r_1 + r_2)}{x+\theta_0^1-\theta_0^2}\right)}{(x+y)(\theta_1^1-\theta_1^2)}, \qquad \beta(x,y) = \frac{((y+\theta_0^2)r_1 + r_2)}{(x+y)(\theta_1^1-\theta_1^2)},$$

where r_1 and r_2 are the right-hand-sides of our linear system on α and β

$$\begin{aligned} r_1 = r_1(x,y) &= \kappa_1\kappa_2 + \kappa_2\kappa_3 + \kappa_3\kappa_1 - (y-\theta_1^2)(x-\theta_0^2) - \theta_0^1(y+\theta_0^2) \\ &\qquad - \theta_1^1(\theta_0^1+\theta_0^2+\theta_1^2), \\ r_2 = r_2(x,y) &= \kappa_1\kappa_2\kappa_3 + \theta_1^1((y-\theta_1^2)(x-\theta_0^2)+\theta_0^1(y+\theta_0^2)). \end{aligned}$$

Schlesinger evolution equations give us the map $\psi : (x,y) \to (\bar{x},\bar{y})$:

$$\begin{cases} \bar{x} = \dfrac{(\alpha-\beta)(\alpha x(\theta_1^1-\theta_1^2) + (1+\theta_0^2)(x(y-\theta_1^2)+y(\theta_0^1-\theta_0^2)))}{(\alpha-\beta)(x(y-\theta_1^2)+(\theta_0^1-\theta_0^2)y) - \alpha(\theta_1^1+1)(\theta_0^1-\theta_0^2)} \\ \bar{y} = \dfrac{(\alpha-\beta)(y(x+\theta_0^1-\theta_0^2)-\theta_1^2 x)}{\alpha(\theta_0^1-\theta_0^2)} \end{cases}.$$

Very complicated! (Finding a simple form for this equation was one of the main motivations behind this project)

The Okamoto surface for the map $\psi : (x, y) \to (\bar{x}, \bar{y})$ is given by the blow-up diagram:

Difference Painlevé Equation of Type d-$P(A_2^{(1)*})$: Schlesinger Transformations

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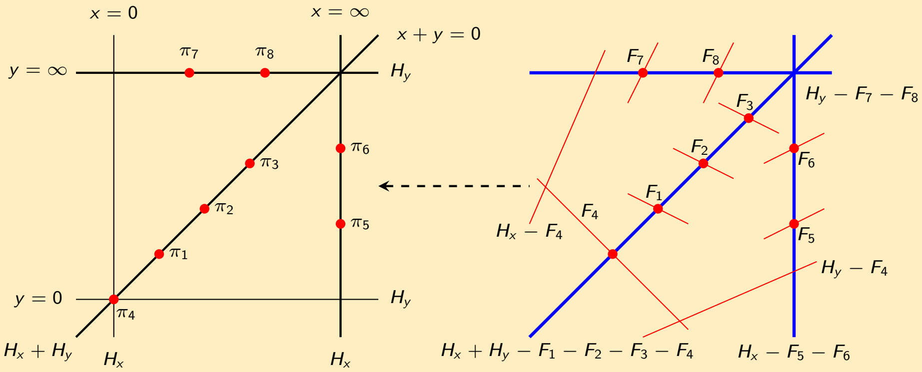

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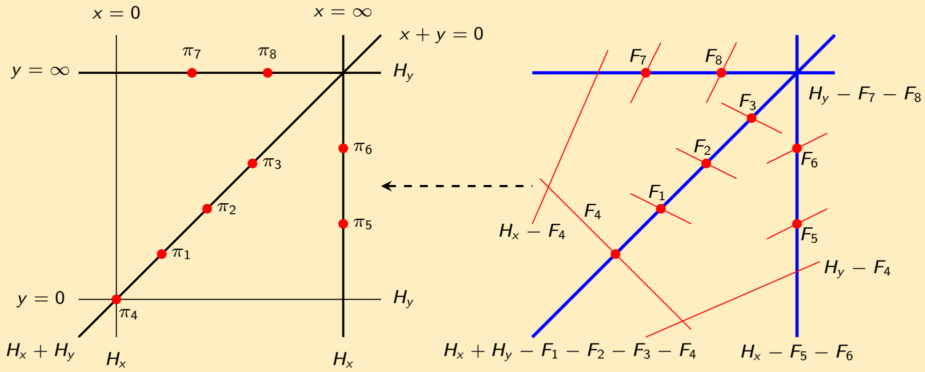

So we see that the configuration structure is the same, but the coordinates of the blowup points are now expressed in terms of the characteristic indices:

$$p_i(\theta_0^2+\kappa_i, -\theta_0^2-\kappa_i), \quad p_4(0,0), \quad p_5\left(\infty, \theta_1^1\right), \quad p_6\left(\infty, \theta_1^2\right), \quad p_7\left(\theta_0^2 - \theta_0^1, \infty\right), \quad p_8\left(\theta_0^2 + 1, \infty\right).$$

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The action of ψ_* on Pic($\mathcal{X}$)

$$
\begin{aligned}
\mathcal{H}_f &\mapsto 2\mathcal{H}_f + 3\mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - 2\mathcal{E}_5 - 2\mathcal{E}_8,\\
\mathcal{H}_g &\mapsto 3\mathcal{H}_f + 5\mathcal{H}_g - 2\mathcal{E}_1 - 2\mathcal{E}_2 - 2\mathcal{E}_3 - 2\mathcal{E}_4 - 3\mathcal{E}_5 - \mathcal{E}_6 - 2\mathcal{E}_8,\\
\mathcal{E}_1 &\mapsto \mathcal{H}_f + 2\mathcal{H}_g - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_8,\\
\mathcal{E}_2 &\mapsto \mathcal{H}_f + 2\mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_8,\\
\mathcal{E}_3 &\mapsto \mathcal{H}_f + 2\mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_8,\\
\mathcal{E}_4 &\mapsto \mathcal{H}_f + 2\mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_5 - \mathcal{E}_8,\\
\mathcal{E}_5 &\mapsto \mathcal{E}_7,\\
\mathcal{E}_6 &\mapsto 2\mathcal{H}_f + 2\mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - 2\mathcal{E}_5 - \mathcal{E}_8,\\
\mathcal{E}_7 &\mapsto 2\mathcal{H}_f + 3\mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - 2\mathcal{E}_5 - \mathcal{E}_6 - 2\mathcal{E}_8,\\
\mathcal{E}_8 &\mapsto \mathcal{H}_g - \mathcal{E}_5,
\end{aligned}
$$

and so the induced action φ_* on the sub-lattice $R^\perp$ is given by the following *translation*:

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \mapsto (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) + (0, 0, 0, -1, 1, 1, -1)\delta,$$

Comparison between different forms of d-$P(\widetilde{A}_2^*)$

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- Compare the parameters and the dynamic on the level of parameters:

$$b_i = \theta_0^2 + \kappa_i,\ b_4 = 0,\ b_5 = \theta_1^1,\ b_6 = \theta_1^2,\ b_7 = \theta_0^1 - \theta_0^2,\ b_8 = -\theta_0^2 - 1.$$

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- This can also be written as follows, with $\delta = \chi(-\mathcal{K}_{\mathcal{X}}) = b_1 + \cdots + b_8 (= -1)$:

$$\varphi : \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix} \mapsto \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 + \delta & b_6 + \delta & b_7 - \delta & b_8 - \delta \end{pmatrix} \qquad \text{deautonomization}$$

$$\psi : \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix} \mapsto \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 - \delta & b_6 & b_7 + \delta & b_8 \end{pmatrix} \qquad \text{Schlesinger Transformations}$$

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- Riemann scheme $\left(\text{which gave d-}P(A_2^{(1)*}) = \Sigma_0(1,3) \circ \left\{ \begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix} \right\} \circ \Sigma_0(1,3) \circ \left\{ \begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix} \right\}.\right)$:

$$\left\{ \begin{matrix} z=0 & z=1 & z=\infty \\ \theta_0^1 & \theta_1^1 & \kappa_1 \\ \theta_0^2 & \theta_1^2 & \kappa_2 \\ 0 & 0 & \kappa_3 \end{matrix} \right\} \overset{\left\{ \begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix} \right\}}{\longmapsto} \left\{ \begin{matrix} z=0 & z=1 & z=\infty \\ \theta_0^1 - 1 & \theta_1^1 + 1 & \kappa_1 \\ \theta_0^2 & \theta_1^2 & \kappa_2 \\ 0 & 0 & \kappa_3 \end{matrix} \right\},$$

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- Translation directions:

$$\varphi_* : (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \mapsto (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) + (0, 0, 0, 1, 0, -1, 0)\delta$$
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Comparison between different forms of d-$P(\widetilde{A}_2^*)$

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- The best approach, however, is through the decomposition. In the same way as we did for φ_*, we can compute and compare the decomposition for ψ_*;

$$\varphi_* = \sigma_r \circ w_5 \circ w_2 \circ w_3 \circ w_6 \circ w_5 \circ w_2 \circ w_4 \circ w_3 \circ w_1 \circ w_2 \circ w_5 \circ w_0 \circ w_1 \circ w_2 \circ w_6 \circ w_5$$
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- This gives us the equivalence!

$$\psi_* = \sigma_r \circ w_1 \circ w_5 \circ \sigma_{r^2} \circ \varphi_* \circ w_5 \circ w_3 = (w_3 \circ w_5) \circ \varphi_* \circ (w_3 \circ w_5)^{-1}$$

Comparison between different forms of d-$P(\widetilde{A}_2^*)$

- Translation directions:

$$\varphi_* : (\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6) \mapsto (\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6) + (0,0,0,1,0,-1,0)\delta$$
$$\psi_* : (\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6) \mapsto (\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6) + (0,0,0,-1,1,1,-1)\delta$$

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- The mapping $w_5 \circ w_3$ gives us the change of variables between the two equations,

$$f = \frac{x(y-\theta_1^1) + y(\theta_0^1+\kappa_1) + (\theta_0^2+\kappa_1)(\theta_0^1+\theta_0^2+\theta_1^1+2\kappa_1)}{y+\theta_0^2+\kappa_1}$$
$$g = \frac{x(y-\theta_0^2-\theta_1^1-\kappa_1) + y(\theta_0^1-\theta_0^2) + (\theta_0^2+\kappa_1)(\theta_0^1+\theta_0^2+2\kappa_1)}{x-\theta_0^2-\kappa_1}$$

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