Special function solutions of Painlevé equations

Alfredo Deaño

University of Kent

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Outline

- Painlevé equations
- Special function solutions
- Examples: $\mathrm{P_{II}}$ and $\mathrm{P_{IV}}$
- Motivation 1: Special function solutions and OPs
- Motivation 2: Complex OPs and numerical quadrature
- Asymptotic analysis as $n \to \infty$
- Other asymptotic regimes
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Painlevé differential equations

Object of interest (after **Painlevé, Gambier, Boutroux...**): nonlinear second order ODEs

$$u''(z) = F(z, u(z), u'(z))$$

with the so-called **Painlevé property**:

Any solution is **free of movable branch points**

Classification: up to changes of variables and integration in terms of (known) special functions, only six new equations remain, the **Painlevé differential equations**.

Painlevé differential equations $\mathrm{P_I} - \mathrm{P_{VI}}$

$$
\begin{aligned}
u'' &= 6u^2 + z, \\
u'' &= zu + 2u^3 + \alpha, \\
u'' &= \frac{(u')^2}{u} - \frac{u'}{z} + \frac{\alpha u^2 + \beta}{z} + \gamma u^3 + \frac{\delta}{u}, \\
u'' &= \frac{(u')^2}{2u} + \frac{3u^3}{2} + 4zu^2 + 2(z^2 - \alpha)u + \frac{\beta}{u}, \\
u'' &= \left(\frac{1}{2u} + \frac{1}{u-1}\right)(u')^2 - \frac{u'}{z} + \frac{(u-1)^2}{z^2}\left(\alpha u + \frac{\beta}{u}\right) + \frac{\gamma u}{z} + \frac{\delta u(u+1)}{u-1}, \\
u'' &= \frac{(u')^2}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-z}\right) - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{u-z}\right)u' \\
&\quad + \frac{u(u-1)(u-z)}{z^2(z-1)^2}\left[\alpha + \frac{\beta z}{u^2} + \frac{\gamma(z-1)}{(u-1)^2} + \frac{\delta z(z-1)}{(u-z)^2}\right].
\end{aligned}
$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$.

Other formulations:

- Symmetric forms (Noumi & Yamada, Forrester & Witte, ...)
- Compatibility between linear systems of equations (Lax pairs):

$$\frac{\mathrm{d}}{\mathrm{d}z}\Psi(z,\lambda) = A(z,\lambda)\Psi(z,\lambda), \qquad \frac{\mathrm{d}}{\mathrm{d}\lambda}\Psi(z,\lambda) = U(z,\lambda)\Psi(z,\lambda),$$

  where $\Psi(z,\lambda) : \mathbb{C}\times\mathbb{C} \mapsto \mathbb{C}^{2\times 2}$.

- Hamiltonian formulation:

$$\frac{\mathrm{d}q}{\mathrm{d}z} = \frac{\partial \mathcal{H}}{\partial p}, \qquad \frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{\partial \mathcal{H}}{\partial p},$$

  for some Hamiltonian $\mathcal{H}(q,p,z,\mathbf{v})$.

Painlevé differential equations $\mathrm{P_I} - \mathrm{P_{VI}}$

- Generic solutions of $\mathrm{P_I} - \mathrm{P_{VI}}$ are often called **Painlevé transcendents**.
- For specific values of the parameters, $\mathrm{P_{II}} - \mathrm{P_{VI}}$ admit important families of solutions: **rational solutions** and **special function solutions**.

Painlevé equations. Special function solutions I

Special function solutions can be obtained as follows:

- Consider the Hamiltonian function $\mathcal{H}(q,p,z,\mathbf{v})$.
- Define a τ function:

$$\mathcal{H}(q,p,z,\mathbf{v}) = \frac{\mathrm{d}}{\mathrm{d}z}\log\tau(z,\mathbf{v}).$$

- Consider Bäcklund transformations

$$T_{\pm}^{n}[\mathcal{H}] = \mathcal{H}[\mathbf{v_0}\pm n\mathbf{v}] = \frac{\mathrm{d}}{\mathrm{d}z}\tau(z,\mathbf{v_0}\pm n\mathbf{v}) = \frac{\mathrm{d}}{\mathrm{d}z}\tau_n(z,\mathbf{v_0})$$

  starting from some initial parameters $\mathbf{v_0}$, and $n\in\mathbb{Z}$.
- Seek a Toda-type equation for τ_n:

$$\mathrm{D}^2\log\tau_n = C_n\frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2},$$

  where D a differential operator and C_n a constant (in z).

- Deduce **Wronskian determinant** forms for the τ-functions (Omakoto 1986, Forrester & Witte 2001):

$$\tau_n(z) = \det \begin{pmatrix} \varphi(z) & \mathrm{D}\varphi(z) & \dots & \mathrm{D}^{(n-1)}\varphi(z) \\ \mathrm{D}\varphi(z) & \mathrm{D}^2\varphi(z) & \dots & \mathrm{D}^{(n)}\varphi(z) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{D}^{(n-1)}\varphi(z) & \mathrm{D}^{(n)}\varphi(z) & \dots & \mathrm{D}^{(2n-1)}\varphi(z) \end{pmatrix}$$

$$= \det\left(\frac{\mathrm{D}^{(j+k)}}{\mathrm{D}z^{j+k}}\varphi(z)\right)_{j,k=0,1,\dots n-1}$$

- The initial values are

$$\tau_0(z) = 1, \qquad \tau_1(z) = \varphi(z),$$

starting with some appropriate **seed function** $\varphi(z)$.

The result are hierarchies of solutions associated with classical special functions:

Painlevé II	Airy functions: $\mathrm{Ai}(z)$, $\mathrm{Bi}(z)$
Painlevé III	Bessel functions: $J_\nu(z)$, $Y_\nu(z)$
Painlevé IV	Parabolic cylinder (Weber) functions: $U(a,z)$ or $D_\nu(z)$
Painlevé V	Kummer (Whittaker) functions: ${}_1F_1(a;c;z)$
Painlevé VI	Gauss hypergeometric functions: ${}_2F_1(a,b;c;z)$

Painlevé II. Half-integer hierarchy

Theorem

The Painlevé II equation has a one-parameter family of solutions in terms of Airy functions if and only if $\alpha = n - \frac{1}{2}$, with $n \in \mathbb{Z}$:

$$u_n(z) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{\tau_{n-1}(z)}{\tau_n(z)}, \qquad n \geq 1,$$

with $\tau_0(z) = 1$ and

$$\tau_n(z) = \det \left(\frac{\mathrm{d}^{j+k}}{\mathrm{d}z^{j+k}} \varphi(z) \right)_{j,k=0,1,\ldots n-1}.$$

The seed function is given by

$$\varphi(z) = C_1 \mathrm{Ai}(-2^{-1/3} z) + C_2 \mathrm{Bi}(-2^{-1/3} z),$$

in terms of classical Airy functions.

Painlevé IV. Weber-Hermite solutions

Theorem

The Painlevé IV equation

$$u'' = \frac{(u')^2}{2u} + \frac{3u^3}{2} + 4zu^2 + 2(z^2-\alpha)u + \frac{\beta}{u}$$

has a one-parameter family of solutions in terms of parabolic cylinder functions if and only if the parameters verify either

$$\beta = -2(2n+1+\varepsilon\alpha)^2,$$

or

$$\beta = -2n^2,$$

with $n \in \mathbb{Z}$. The seed functions are solutions of

$$\frac{\mathrm{d}^2\varphi_\nu}{\mathrm{d}z^2} - 2\varepsilon z\frac{\mathrm{d}\varphi_\nu}{\mathrm{d}z} + 2\varepsilon\nu\varphi_\nu = 0, \qquad \varepsilon^2 = 1, \quad \nu = -(1+\varepsilon\alpha).$$

Motivation 1: Special function solutions and OPs

Special function solutions of Painlevé equations appear when we consider **semiclassical orthogonal polynomials** (OPs).

One important case is given by OPs with respect to

$$w(x;t) = w_0(x)W(x,t),$$

where $w_0(x)$ is a classical weight and $t \in \mathbb{R}$. We have

$$\int_a^b p_n(x;t)x^k w(x;t)\mathrm{d}x = 0, \qquad k = 0, 1, \ldots, n-1,$$

and the recurrence relation

$$xp_n(x;t) = p_{n+1}(x;t) + \alpha_n(t)p_n(x;t) + \beta_n(t)p_{n-1}(x;t).$$

Motivation 1: Special function solutions and OPs

- Basor, Chen & Ehrhardt (2009):

$$w(x,t) = (1-x)^{\alpha}(1+x)^{\beta}\mathrm{e}^{-tx}, \qquad \alpha,\beta > -1, \quad x \in [-1,1],$$

  related to Painlevé V.

- Chen & Its (2010):

$$w(x,t) = x^{\alpha}\mathrm{e}^{-x-t/x}, \qquad \alpha > -1, \quad x \in [0,\infty),$$

  related to Painlevé III.

- Van Assche, Filipuk & Zhang (2014):

$$w(x,t) = \mathrm{e}^{-x^3+tx}, \qquad x \in \Gamma \subset \mathbb{C},$$

  related to Painlevé II.

- Boelen & Van Assche (2010), Clarkson & Jordaan (2014):

$$w(x,t) = x^{\lambda}\mathrm{e}^{-x^2+tx}, \qquad \lambda > -1, \quad x \in [0,\infty),$$

  related to Painlevé IV.

What do you mean by 'related to Painlevé...'?

One considers **deformation equations** for $\alpha_n(t)$ and $\beta_n(t)$:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\alpha_n = F(\alpha_{n\pm k}, \beta_{n\pm k}), \\ \frac{\mathrm{d}}{\mathrm{d}t}\beta_n = G(\alpha_{n\pm k}, \beta_{n\pm k}). \end{cases}$$

These equations are often 'interesting' in integrable systems (Toda lattice...).

Together with string (Freud) equations, one often finds equations of Painlevé type.

Important observations

- Solutions of Painlevé are sometimes not directly the recurrence coefficients, but functions of these, with extra terms and/or changes of variable.

  For example (after Clarkson & Jordaan), if

  $$w(x;t) = x^{\lambda} \mathrm{e}^{-x^2+tx}, \qquad \lambda > -1,$$

  then the function

  $$q_n(z) = 2\alpha_n(t) + t,$$

  with $z = \frac{t}{2}$ satisfies Painlevé IV in z.
- The solutions obtained from deforming OPs are typically not 'generic', but with very specific values of the parameters.

In the study of oscillatory integrals

$$I[f] = \int_a^b f(x) \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x, \qquad \omega \gg 1,$$

direct Gaussian quadrature is not efficient or reliable.

One alternative is to deform $[a,b]$ into the complex plane along paths of **steepest descent** and then integrate.

An interesting example: $g(x) = x^3 - cx$, with $c \in (0,1)$.

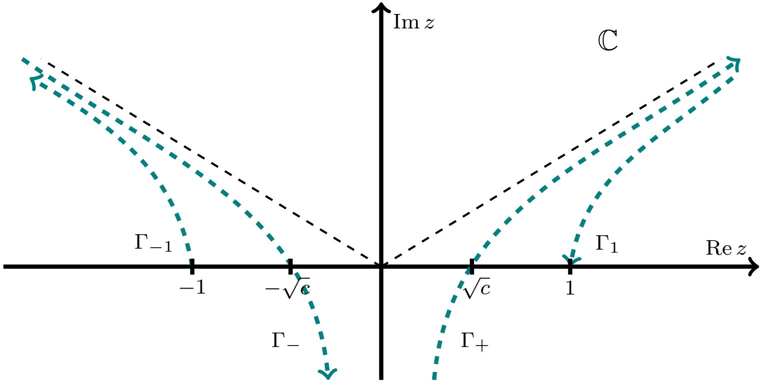

Figure: Steepest descent paths for the oscillator $g(x) = \frac{x^3}{3} - cx$.

Motivation 2: Complex OPs and numerical quadrature

- Standard approach: parametrize the paths, discretise and use Gauss–Laguerre or Gauss–Hermite quadrature.
- Not so standard approach: can we construct **one** quadrature rule for the path $\Gamma := \Gamma_+ \cup \Gamma_-$?

Motivation 2: Complex OPs and numerical quadrature

- This idea leads to consider (formal, complex, non-Hermitian) OPs defined as

$$\int_\Gamma p_n^\omega(z) z^k \mathrm{e}^{\mathrm{i}\omega(\frac{z^3}{3}-cz)} \mathrm{d}z = 0, \qquad z = 0, 1, \ldots, n-1.$$

- Does $p_n^\omega(z)$ exist for any n and ω?
- We can study the Hankel determinant

$$H_n(\omega) = \det\left(\mu_{j+k}(\omega)\right)_{j,k=0}^{n-1}, \qquad \mu_m(\omega) = \int_\Gamma z^m \mathrm{e}^{\mathrm{i}\omega(\frac{z^3}{3}-cz)} \mathrm{d}z,$$

and try to determine regimes for which $H_n(\omega) \neq 0$.

Some references: D., Huybrechs & Kuijlaars 2010, Huybrechs, Kuijlaars & Lejon 2013, 2017?

Asymptotic analysis

In some cases, the tau functions

$$\tau_n(z) = \det \left(\frac{\mathrm{D}^{(j+k)}}{\mathrm{D}z^{j+k}} \varphi(z) \right)_{j,k=0,1,\ldots n-1}$$

can be analysed asymptotically as $n \to \infty$:

- Match the seed function and its derivatives with the moments of a suitable weight function.
- Rewrite the Wronskian determinant as a Hankel determinant H_n for this new weight function.
- Use the machinery of asymptotics of OPs (Riemann–Hilbert or other) to obtain asymptotics for H_n.

Similar ideas for rational solutions by Bertola, Bothner (2015), Balogh, Bertola, Bothner (2016), Buckingham (2017).

Asymptotic analysis for P_{IV}

Consider the seed function for special function solutions of P_{IV}:

$$\varphi_\nu(z) = \left\{C_1 U\left(-\nu - \tfrac{1}{2}, \sqrt{2}\, z\right) + C_2 U\left(-\nu - \tfrac{1}{2}, -\sqrt{2}\, z\right)\right\} \mathrm{e}^{\frac{1}{2} z^2},$$

for $\nu \notin \mathbb{Z}$, and the Wronskian

$$\tau_{n,\nu}(v) = \det\left(\frac{\mathrm{d}^{(j+k)}}{\mathrm{d}z^{j+k}} \varphi_\nu(z)\right)_{j,k=0,1,\ldots n-1}$$

We will study the asymptotic behavior as $n \to \infty$, coupled with z.

If we have the weight

$$w(x;v,s,\alpha)=\mathrm{e}^{-x^2}|x-v|^{\alpha}\begin{cases} s, & \text{if } x<v,\\ 1, & \text{if } x>v,\end{cases}$$

with $v\in\mathbb{R}$, $s\in[0,1]$, $\mathrm{Re}\,\alpha>-1$, its Hankel determinant satisfy

$$H_n(v,s,\alpha)=\mathrm{e}^{-nv^2}\Gamma(1+\alpha)^n 2^{-n^2-\frac{n(\alpha-1)}{2}}\tau_{n,-\alpha-1}(v),$$

where $C_1=1$ and $C_2=s$.

Asymptotic analysis for P_{IV}

In the paper

C. Charlier, A. D., Asymptotics for Hankel determinants associated to a Hermite weight with a varying discontinuity, *arXiv:1708.02519*

the authors obtain

- Asymptotics as $n \to \infty$ of $H_n(v, 0, \alpha)$
- Asymptotics as $n \to \infty$ of $H_n(v, s, \alpha)$, with $s \in (0, 1]$.
- Uniform asymptotics in s.

Tools:

- OPs with respect to the weight $w(x; v, s, \alpha)$.
- Deformation equations in the parameters v and s.

Results

Theorem

Let $t \in (-1,1)$ and $\operatorname{Re}\alpha > -1$, then as $n \to \infty$

$$\log \frac{H_n(\sqrt{2n}t, 0, \alpha)}{H_n(0,0,\alpha)} = C_1(t)n^2 + C_2(t,\alpha)n + C_3(t,\alpha) + \mathcal{O}(n^{-1}),$$

where $C_1(t)$, $C_2(t,\alpha)$ and $C_3(t,\alpha)$ are explicit, and the error term is uniform for t in compact subsets of $(-1,1)$.

Large n asymptotics of $H_n(0,0,\alpha)$ can be obtained from A. D., Nicholas J. Simm. (2017), and also Krasovsky (2007).

Results

Theorem

If $t \in (-1,1)$ and $s \in (0,1]$, then as $n \to \infty$

$$\log \frac{H_n(\sqrt{2n}t, s, \alpha)}{H_n(\sqrt{2n}t, 1, \alpha)} = \left[2\arcsin t + 2t\sqrt{1-t^2} + \pi\right] \frac{\log s}{2\pi} n + \frac{(\log s)^2}{4\pi^2} \log n + \tilde{c}_0 + \mathcal{O}\left(\frac{\log n}{n}\right),$$

where the constant $\tilde{c}_0 = \tilde{c}_0(t, s, \alpha)$ is explicit.

Results

To obtain uniform asymptotics, we write

$$s = \mathrm{e}^{-\lambda n}, \qquad \lambda \geq 0,$$

and we use a (explicit) critical value $\lambda_c(t)$.

Theorem

If $t \in (-1,1)$ and $\operatorname{Re}\alpha > -1$ and $\lambda \geq \lambda_c(t)$, then as $n \to \infty$

$$\log \frac{H_n(\sqrt{2n}t, \mathrm{e}^{-\lambda n}, \alpha)}{H_n(\sqrt{2n}t, 0, \alpha)} = \mathcal{O}\left(n^{-1/2}\mathrm{e}^{-n(\lambda-\lambda_c(t))}\right),$$

and large n asymptotics for $\log H_n(\sqrt{2n}t, 0, \alpha)$ are given by the previous theorem.

Results

Theorem

If $0 \leq \lambda \leq \lambda_c(t)$, then as $n \to \infty$

$$\lim_{n\to\infty} \frac{1}{n^2} \log \frac{H_n(\sqrt{2n}t, \mathrm{e}^{-\lambda n}, \alpha)}{H_n(\sqrt{2n}t, 1, \alpha)} = -\int_0^{\lambda} \Omega(t, \tilde{\lambda})\mathrm{d}\tilde{\lambda},$$

where

$$\Omega(t,\lambda) = \frac{2}{\pi}\int_a^b \rho(x;t,\lambda)\mathrm{d}x, \quad \rho(x;t,\lambda) = \sqrt{c-x}\sqrt{\frac{x-b}{x-t}}\sqrt{x-a},$$

and $a < b < t < c$ (depending on λ) are given by

$$\begin{aligned} t &= a+b+c, \\ 2 &= a^2+b^2+c^2-t^2, \\ \lambda &= 4\int_b^t \rho(x;t,\lambda)\mathrm{d}x. \end{aligned}$$

See also ideas in

- C. Charlier and T. Claeys, Thinning and conditioning of the Circular Unitary Ensemble, *Random Matrices Theory Appl.* **6** *(2017).*
- C. Charlier. Asymptotics of Hankel determinants with a one-cut regular potential and Fisher–Hartwig singularities. *arXiv:1706.03579*
- I. Krasovsky. Correlations of the characteristic polynomials in the Gaussian Unitary Ensemble or a singular Hankel determinant. *Duke Math. J.* **139** *(2007) 581–619.*

Painlevé II and a cubic model

A seed function for special function solutions of Painlevé II is

$$\varphi(z) = C_1 \mathrm{Ai}(-2^{-1/3} z) + C_2 \mathrm{Bi}(-2^{-1/3} z).$$

In order to study the tau functions

$$\tau_n(z) = \det \left(\frac{\mathrm{d}^{j+k}}{\mathrm{d}z^{j+k}} \varphi(z) \right)_{j,k=0,1,\ldots n-1},$$

we can consider (complex) OPs with respect to the weight

$$w(\zeta;t) = \mathrm{e}^{\frac{\zeta^3}{3} - t\zeta}$$

on a suitable contour.

Painlevé II and a cubic model

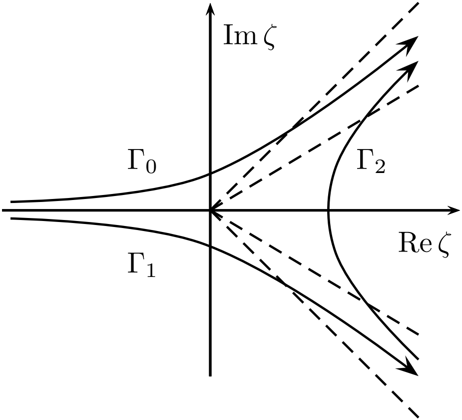

Figure: Contour for the construction of the OPs. A general path would be $\Gamma = \alpha\Gamma_0 + \beta\Gamma_1$.

Painlevé II and a cubic model

The Hankel determinant H_n is related to the partition function of a cubic Hermitian random matrix model.

See Bleher, D., 2010, 2013, and also similar cases in Bertola, Tobvis, 2011, 2016.

This cubic model has been studied in D., Huybrechs & Kuijlaars 2010 (case $t = 0$), see also Huybrechs, Kuijlaars & Lejon 2013 and Álvarez, Martínez Alonso & Medina 2013-15.

General complex t, one-cut case, Bleher, D. & Yattselev, 2016.

In

P. A. Clarkson. On Airy solutions of the second Painlevé equation. *Stud. Appl. Math., 137 (2016), 93–109.*

the author gives determinant expressions for solutions of $\mathrm{P_{II}}$ and $\mathrm{S_{II}}$ (symmetric form), as well as large z asymptotics.

Taking the seed function

$$\varphi(z;\vartheta)=\cos(\vartheta)\mathrm{Ai}(-2^{-1/3}z)+\sin(\vartheta)\mathrm{Bi}(-2^{-1/3}z),$$

different asymptotic behavior as $z\to-\infty$ is found depending on ϑ:

$$q_n(z;\vartheta)=\frac{\mathrm{d}}{\mathrm{d}z}\log\frac{\tau_{n-1}(z;\vartheta)}{\tau_n(z;\vartheta)}=\begin{cases}-\sqrt{-\dfrac{z}{2}}+\mathcal{O}(z^{-1}), & \vartheta=0,\\ \sqrt{-\dfrac{z}{2}}+\mathcal{O}(z^{-1}), & \vartheta\neq 0.\end{cases}$$

Via Stokes multipliers (Flaschka & Newell (1980))

$$s_1 = s_3 = -s_2 = (-1)^{n+1}, \qquad \alpha = -n - \frac{1}{2}.$$

Other references:

- Fokas, Its, Kapaev, Novokshenov (2006):

$$q_n(z;0) = -\sqrt{-\frac{z}{2}} + \mathcal{O}(z^{-2/5}), \qquad \arg z \in \left[\frac{2\pi}{3}, \pi\right].$$

- Kuijlaars, Its, Östensson (2008). Asymptotic behavior valid (as tronquée solutions) for $\arg z \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$

Other questions

- Are there tronquée solutions for certain choices of the seed function?
- Structure of pole fields in $\mathbb{C}$? Fornberg & Weideman (2013).
- Monodromy data for special function solutions?
- Numerical computation?
  - Boundary value problem with Padé approximations: Fornberg & Weideman, Fornberg & Reeger, Fasondini, Fornberg & Weideman.
  - Numerical solution of Riemann–Hilbert problems: Olver & Trogdon.
  - Fredholm determinants: Bornemann.
- ...

That's all for now

Thank you for your attention!!