Tronquée Solutions of Painlevé equations

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Painlevé equations and Applications. Workshop in memory of A.A. Kapaev
Generic solutions of the Painlevé equations have singularities (poles) in any sector towards $\infty$ in the complex plane. But some are free of poles for large $z$ in sectors:

**P$_1$**: $u'' = 6u^2 - z$  

*Boutrox*:  
- there is a one-parameter family with no poles at infinity in a two adjacent sectors bd, by $\arg z = \frac{2k\pi i}{5}$, *tronquéee solutions*.  
- There are solutions with no poles in four such sectors: *tritronquéée*

*Dubrovin’s conjecture (2009)*: tonquéée have no poles in the sector.  
*Proved so* by O. Costin, H. Huang, S. Tanveer (2015)

**P$_{II}$**: $u'' = zu + 2u^3 + \alpha$  

*Boutrox*: sectors bd. by $\arg z = \frac{2k\pi i}{6}$  
Existence of tronquéee solutions proved by N. Joshi and M. Mazzocco (2003)

*Novokshenov’s conjecture (2014)*: tonquéée of any $P_n$ have no poles in the sector.  
*Proved so* for the Hastings-McLeod solution (2-tronqée) by M. Huang, S-X Xu, Lun Zhang (2015).
$P_{\text{III}}, P_{\text{IV}}$

- Existence of tronquée & position of the first array of poles, X. Xia (submitted)

$P_{\text{V}}$

Andreev, Kitaev - the exp small correction (1997)
S. Shimomura (at $w = 1$) (2011)

A. Kapaev’s work on tritronquée solutions, including he calculated the exp small term for $P_{\text{I}}$

Methods: classical analysis, isomonodromy methods.
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Methods: classical analysis, isomonodromy methods.
Our approach (RDC, OC, MH, S. Tanveer, Xiaoyue Xia):

We are studying truncated solutions using generalized Borel summation of their transseries (multi-instanton expansions).

Advantages:

- Once the equation is normalized (∃ almost algorithmic procedures)
- ...there are general theorems establishing

\[ \{ \text{transseries solutions, in sectors} \} \leftrightarrow \{ \text{actual solutions an. there to } \infty \} \]

- The location of the first array of poles, beyond the sector of analyticity is also obtained.
New results with these techniques

- Combined with novel methods in the finite plane, Dubrovin’s conjecture was proved.
- New (readable!) proof for $P_1$ is coming soon. (OC, RDC)
- Combined with novel methods in the finite plane, Novokshenko’s conjecture was proved for McLeod solution of $P_2$.
- Obtained the Stokes multiplier directly for $P_1$. (OC, RDC, MH, 2015)
- Location of first arrays of poles for $P_3, P_4$ (X.X.)

NEW:

- Combined with a new type of convergent expansions $\mapsto$ very efficient numerical methods to calculate solutions of $P_n$. (O Costin, G. Dunne)
- I will present some
  - recent work on $P_I$ (OC, RDC)
  - ongoing work on $P_5$ (RDC)
Example: linear equation $y' + y = x^{-2}$

Point $x = \infty$ is a rank 1 irregular sing. pt.

Has unique power series sol as $x \to \infty$: $\tilde{y}_0(x) = \sum_{n=2}^{\infty} \frac{(n-1)!}{x^n}$. Divergent.

General solution:
$$y(x; C) = y_0(x) + Ce^{-x}, \quad y_0(x) = e^{-x} \int_x^{\infty} e^s s^{-2} ds \sim \tilde{y}_0(x) \quad (x \to \infty)$$

Phenomena at irreg. sing. pts.:
- power series sol are divergent
- loss of information (1-param. fam. of sol. asy. to the same series)
- asymptoticity holds only in sectors (here for $|\arg x| < \pi/2$)

Natural: complete formal solution $\tilde{y}(x) = \tilde{y}_0(x) + Ce^{-x}$ for $x \to +\infty$. 
Intro to transseries expansions (multi-instantons)

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Eq. $y' + y = \frac{1}{x^2}$ has
the complete formal solution $\tilde{y}(x) = \tilde{y}_0(x) + Ce^{-x}$ for $x \to +\infty$

Simplest example of transseries.

Features:
- It satisfies the equation where it originated.
- It is not an asymptotic expansion à la Poincaré ($Ce^{-x} \ll x^{-n} \forall n$),
- but it is well ordered for $|\arg x| < \pi/2$ (terms are decreasing).
- It contains the free parameter $C$. 
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Nonlinear example: \[ y' + y = \frac{1}{x^2} + y^4 \]

Unique power series sol: \( \tilde{y}_0(x) = \frac{1}{x^2} + \frac{2}{x^3} + \frac{6}{x^4} + \ldots \ (x \to \infty) \)
Divergent.

Smaller terms? Substitute \( y = \tilde{y}_0(x) + g(x) \) assuming \( g(x) \ll x^{-n} \forall n \).
Obtain: \( \tilde{F}_0(z) + \tilde{F}_1(z)g(z) + g'(z) + [\text{Quadratic in } g] = 0 \)

\[ \tilde{y}_0(x) \text{ is formal solution} \implies \tilde{F}_0(z) = 0. \]

\[ \text{Quadratic } \ll g \implies \text{neglect.} \]

\[ \tilde{y}_0(x) \sim 4\tilde{y}_0^3g \implies g(x) \sim Ce^{-x}\tilde{y}_1(x). \]

Only \( \tilde{y}_0(x) + Ce^{-x}\tilde{y}_1(x) \) is not an exact sol. Look for even smaller terms.
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Only  \( \tilde{y}_0(x) + Ce^{-x}\tilde{y}_1(x) \) is not an exact sol. Look for even smaller terms.
A complete formal solution:

\[ \tilde{y}(x) = \tilde{y}_0(x) + C e^{-x} \tilde{y}_1(x) + C^2 e^{-2x} \tilde{y}_2(x) + C^3 e^{-3x} \tilde{y}_3(x) + \ldots \]

where \( \tilde{y}_n(x) \) are divergent power series and \( C \) is a free parameter.

**General 1-dim**

\[
y' + \left( \lambda - \frac{\alpha}{x} \right) y = g(x^{-1}, y), \quad g = O(x^{-2}) + O(y^2) \quad (x \to \infty, y \to 0)
\]

has a complete formal solution (in the RHP)

\[
\tilde{y}(x) = \tilde{y}_0(x) + C e^{-\lambda x} \tilde{y}_1(x) + C^2 e^{-2\lambda x} \tilde{y}_2(x) + C^3 e^{-3\lambda x} \tilde{y}_3(x) + \ldots
\]

where \( \tilde{y}_n(x) = x^{n\alpha} \tilde{s}_n(x) \), where \( \tilde{s}_n(x) \) are integer power series.

Valid for \( x \to \infty \) with \( |\arg(\lambda x)| < \pi/2 \).
Systems with $x = \infty$ a rank 1 irregular singularity:

**Normal form:**

$$y' + \left( \Lambda - \frac{1}{x} A \right) y = g(x^{-1}, y)$$

with $g$ analytic at $(0, 0)$, with $g(x^{-1}, y) = O(x^{-2}) + O(|y|^2)$

If $\Lambda, A$ are diagonalizable, then

$$\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_d), \ A = \text{diag}(\alpha_1, \ldots, \alpha_d)$$

(nonresonant) have formal solution:

$$\tilde{y} = \tilde{y}(x; C) = \tilde{y}_0(x) + \sum_{k \in \mathbb{N}^d \setminus 0} C^k e^{-\lambda \cdot k x} \tilde{y}_k(x)$$

where $\tilde{y}_k(x) = x^{\alpha \cdot k} \tilde{s}_k(x)$ are power series (divergent), determined algorithmically.

(If resonant - also logs.)

Note: the free parameters are beyond all orders of the unique power series.
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Note: the free parameters are beyond all orders of the unique power series.
Systems with $\infty$ a rank 1 singularity $y' + \left(\Lambda - \frac{1}{x}A\right)y = g(x^{-1}, y)$ have formal solutions:

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Introduced by Fabris (1885). Studied by Cope (1934).

Vastly generalized by Ecalle (1981) to formal expressions closed under all operations.

In logic. More recently, in physics (QFT, in particular).
So: solutions $y \to 0$ as $x \to \infty$ have a unique asymptotic power series:

$$y \sim \tilde{y}_0 \equiv \sum_{n=0}^{+\infty} x^{-n} y_{0,n}$$ (usually divergent)

Conversely:

**Theorem** [Wasow]

If $\tilde{y}_0$ formally solves $y' + \left( \Lambda - \frac{1}{x} A \right) y = g(x^{-1}, y)$ (and a nonres. cond. on $\lambda_j, \alpha_j$) then there exists a true solution

$$y \sim \hat{y}_0, \quad |x| > R, \quad x \in S = \text{some sector opening} < \pi$$

Usually, there are many such solutions.

Recall that the free parameters are beyond all orders of $\tilde{y}_0$...
Correspondence between formal and actual solutions:

**Example:** linear equation \( y' + y = x^{-2} \)

Formal solution \( \tilde{y}(x; C) = \sum_{n \geq 2} (n - 1)! x^{-n} + Ce^{-x} \)

Since \( \mathcal{L}(p^{n-1}) = (n - 1)!x^{-n} \) it is natural to attempt \( \tilde{y}_0 = \mathcal{L}(\text{function}) \).

Recall: Borel transform is formal \( \mathcal{L}^{-1}: B(x^{-\alpha}) = p^{\alpha-1}/\Gamma(\alpha) \) for \( \alpha > 0 \).

Let \( y(x) = \mathcal{L}_\phi Y(x) := \int_{e^{i\phi} \mathbb{R}_+} e^{-px} Y(p) dp \) eq. becomes \( Y(p) = \frac{p}{1 - p} \).

We can integrate for \( \phi \neq 0 \), obtaining:

\[
\begin{align*}
y_0^+(x) &= \mathcal{L}_\phi Y(x) \quad \text{for } -\phi = \arg x \in \left(0, \frac{\pi}{2}\right) \\
y_0^-(x) &= \mathcal{L}_\phi Y(x) \quad \text{for } -\phi = \arg x \in \left(-\frac{\pi}{2}, 0\right)
\end{align*}
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Correspondence between formal and actual solutions: Generalized Borel Summation (O Costin 1995, '98)

**Example:** linear equation $y' + y = x^{-2}$

Formal solution $\tilde{y}(x; C) = \sum_{n \geq 2} (n - 1)! x^{-n} + Ce^{-x}$

Since $L(p^{n-1}) = (n - 1)! x^{-n}$ it is natural to attempt $\tilde{y}_0 = L(function)$.

Recall: Borel transform is formal $L^{-1}$: $B(x^{-\alpha}) = p^{\alpha-1}/\Gamma(\alpha)$ for $\alpha > 0$.

Let $y(x) = L_\phi Y(x) := \int_{e^{i\phi} \mathbb{R}_+} e^{-px} Y(p) dp$ eq. becomes $Y(p) = \frac{p}{1 - p}$.

We can integrate for $\phi \neq 0$, obtaining:

$$y_0^+(x) = L_\phi Y(x) \quad \text{for} \quad -\phi = \arg x \in \left(0, \frac{\pi}{2}\right)$$

and

$$y_0^-(x) = L_\phi Y(x) \quad \text{for} \quad -\phi = \arg x \in \left(-\frac{\pi}{2}, 0\right)$$
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Example: linear equation \( y' + y = x^{-2} \)

Formal solution \( \tilde{y}(x; C) = \sum_{n \geq 2} (n - 1)! \, x^{-n} + Ce^{-x} \)

Since \( \mathcal{L}(p^{n-1}) = (n - 1)!x^{-n} \) it is natural to attempt \( \tilde{y}_0 = \mathcal{L}(\text{function}) \).

Recall: Borel transform is formal \( \mathcal{L}^{-1}: B(x^{-\alpha}) = p^{\alpha-1}/\Gamma(\alpha) \) for \( \alpha > 0 \).

Let

\[
y(x) = \mathcal{L}_\phi Y(x) := \int_{e^{i\phi} \mathbb{R}^+} e^{-px} Y(p) dp
\]

eq. becomes \( Y(p) = \frac{p}{1 - p} \).

We can integrate for \( \phi \neq 0 \), obtaining:

\[
y_0^+(x) = \mathcal{L}_\phi Y(x) \quad \text{for} \quad -\phi = \arg x \in \left(0, \frac{\pi}{2}\right)
\]

and

\[
y_0^-(x) = \mathcal{L}_\phi Y(x) \quad \text{for} \quad -\phi = \arg x \in \left(-\frac{\pi}{2}, 0\right)
\]
**Example:** linear equation $y' + y = x^{-2}$

Formal solution $\tilde{y}(x; C) = \sum_{n \geq 2} (n - 1)! x^{-n} + C e^{-x}$

Since $\mathcal{L}(p^{n-1}) = (n - 1)! x^{-n}$ it is natural to attempt $\tilde{y}_0 = \mathcal{L}(\text{function})$.

Recall: Borel transform is formal $\mathcal{L}^{-1}: B(x^{-\alpha}) = p^{\alpha-1}/\Gamma(\alpha)$ for $\alpha > 0$.

Let $y(x) = \mathcal{L}_\phi Y(x) := \int_{e^{i\phi} \mathbb{R}_+} e^{-px} Y(p) dp$ eq. becomes $Y(p) = \frac{p}{1 - p}$.

We can integrate for $\phi \neq 0$, obtaining:

$$y_0^+(x) = \mathcal{L}_\phi Y(x) \quad \text{for} \quad -\phi = \arg x \in \left(0, \frac{\pi}{2}\right)$$

and

$$y_0^-(x) = \mathcal{L}_\phi Y(x) \quad \text{for} \quad -\phi = \arg x \in \left(-\frac{\pi}{2}, 0\right)$$
\( y_0^+(x) = \mathcal{L}_\phi Y(x) \) for \( -\phi = \arg x \in (0, \frac{\pi}{2}) \)

- does not depend on \( \phi \)
- can be analytically continued in the RHP (and beyond)

Same for \( y_0^-(x) = \mathcal{L}_\phi Y(x) \) for \( -\phi = \arg x \in (-\frac{\pi}{2}, 0) \).

In general \( y_0^+(x) \neq y_0^-(x) \).

In fact \( \frac{1}{2\pi i} \left[ y_0^+(x) - y_0^-(x) \right] = e^{-x} \) recovers the small exponential.
\[ y_0^+(x) = L_\phi Y(x) \quad \text{for } -\phi = \arg x \in (0, \frac{\pi}{2}) \]

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In fact \( \frac{1}{2\pi i} [y_0^+(x) - y_0^-(x)] = e^{-x} \) recovers the small exponential.
**Example:** nonlinear equation \[ y' + y = x^{-2} + y^4 \] with formal sol.

\[ \tilde{y}(x) = \tilde{y}_0(x) + \sum_{n \geq 1} C^n e^{-nx} \tilde{y}_n(x) \]

Take \( L^{-1} \implies (1 - p) Y(p) = 1 + Y^4(p) \)

- Clearly \( \exists! \) solution \( Y_0(p) \) analytic at \( p = 0 \). It is analytic for \( |p| < 1 \).
- Clearly \( Y_0(p) \) is singular at \( p = 1 \).
- Convolution \( \tilde{\to} \) the singularity at \( p = 1 \) gives rise to singularities at \( p = 2, 3, 4, \ldots \) (an array, equally spaced).

Let \( y_0^+(x) = L_{\phi} Y_0(x) \) for \( -\phi = \arg x \in (0, \frac{\pi}{2}) \). Similarly \( y_0^-(x) \).

Summation of other \( \tilde{y}_n \): similarly, let \( y_n = L_{\phi} B \tilde{y}_n \). 
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$\tilde{y}(x) = \tilde{y}_0(x) + \sum_{n \geq 1} C^n e^{-nx} \tilde{y}_n(x)$

Take $L^{-1} \implies (1 - p) Y(p) = 1 + Y^*(4)(p)$

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- Convolution $\leadsto$ the singularity at $p = 1$ gives rise to singularities at $p = 2, 3, 4, \ldots$ (an array, equally spaced).

Let $y_0^+(x) = L_\phi Y_0(x)$ for $-\phi = \text{arg } x \in (0, \frac{\pi}{2})$. Similarly $y_0^-(x)$.

Summation of other $\tilde{y}_n$: similarly, let $y_n = L_\phi B \tilde{y}_n$. 
Example: nonlinear equation $y' + y = x^{-2} + y^4$ with formal sol.

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**Example:** nonlinear equation \( y' + y = x^{-2} + y^4 \) with formal sol.

\[ \tilde{y}(x) = \tilde{y}_0(x) + \sum_{n \geq 1} C^n e^{-nx} \tilde{y}_n(x) \]

Take \( \mathcal{L}^{-1} \Rightarrow (1 - p) Y(p) = 1 + Y^4(p) \)

- Clearly \( \exists! \) solution \( Y_0(p) \) analytic at \( p = 0 \). It is analytic for \( |p| < 1 \).
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Summation of other \( \tilde{y}_n \): similarly, let \( y_n = \mathcal{L}_\phi B \tilde{y}_n \).
Example: nonlinear equation \( y' + y = x^{-2} + y^4 \) with formal sol.
\[
\ddot{y}(x) = \ddot{y}_0(x) + \sum_{n \geq 1} C^n e^{-nx} \tilde{y}_n(x)
\]

Take \( \mathcal{L}^{-1} \implies (1 - p) Y(p) = 1 + Y^4(p) \)

- Clearly \( \exists! \) solution \( Y_0(p) \) analytic at \( p = 0 \). It is analytic for \( |p| < 1 \).
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- Convolution \( \sim \) the singularity at \( p = 1 \) gives rise to singularities at \( p = 2, 3, 4, \ldots \) (an array, equally spaced).

Let \( y^+_0(x) = \mathcal{L}_\phi Y_0(x) \) for \( -\phi = \arg x \in (0, \frac{\pi}{2}) \). Similarly \( y^-_0(x) \).

Summation of other \( \ddot{y}_n \): similarly, let \( y_n = \mathcal{L}_\phi B \tilde{y}_n \).
General result

**Theorem** (O. Costin, 1998)

Let \( \frac{d}{dx} y + (\Lambda - \frac{1}{x} A) y = g(x^{-1}, y) \) nonres., \( \Lambda, A \) diag., \( g \) analytic at \((0, 0)\).

Any formal solution \( \tilde{y} = \tilde{y}(x; C) = \tilde{y}_0(x) + \sum_{k \in \mathbb{N}^d \setminus 0} C^k e^{-\lambda \cdot kx} \tilde{y}_k(x) \)

can be summed along any direction \textbf{not an antistokes lines} \((\pm i\lambda_j \mathbb{R}_+)\)

The series \( y = y_0(x) + \sum_{k \in \mathbb{N}^d \setminus 0} C^k e^{-\lambda \cdot kx} y_k(x) \) converges for large \( |x| \)
to an actual solution analytic on open sector bounded by two consecutive antistokes lines.

Conversely, any such solutions is a summation of a transseries.

Also: **RESURGENCE** all the \( y_n \) can be recovered from \( y_0 \):

\[
y_n(x) = \sum_j \alpha_{j,n} \int_{d_{j,n}} e^{-px} Y_0(p) \, dp
\]

balanced averages of Laplace transforms along \textbf{paths winding in prescribed ways} among \( p = k\lambda j \).
(1957-59) Iwano showed that $y(x; \mathbf{C}) = y_0(x) + \sum C^k e^{-\lambda \cdot k x \cdot \alpha \cdot k} y_k(x)$ with $y_k(x)$ analytic, and convergent in sectors.

(1981) Ecalle constructed the summation of transseries (formal solutions of most problems), establishing an isomorphism with a class of functions ("analyzable").


(1998) O. Costin proved generalized Borel summation for transseries solutions of rank 1, their 1-to-1 correspondence with solutions $y(x) \to 0$ (in a sector), and compatibility with all operations.

(2001-04) Braaksma proved similar results for solutions of difference equations.
Near the boundary of the sector of analyticity

Solutions \( y(x; C) \to 0 \) for \( x \to \infty, \; x \in d \) are analytic in \( S \) for \( |x| \) large. **Question:** what happens to \( y(x; C) \) as \( x \) approaches \( \partial S \)?

**Example:** \( d=1 \)

\[
y' + \left( 1 - \frac{\alpha}{x} \right) y = g(x^{-1}, y) \quad (\lambda = 1).\]

Formal solution:

\[
\tilde{y}(x; C) = \hat{y}_0(x) + C e^{-x} x^\alpha \tilde{s}_1(x) + C^2 e^{-2x} x^{2\alpha} \tilde{s}_2(x) + C^3 e^{-3x} x^{3\alpha} \tilde{s}_3(x) + \ldots
\]

with \( \tilde{s}_k(x) = \sum_{j=0}^{\infty} \frac{y_{k,j}}{x^j} \) valid in the sector \( S_{\text{trans}} = \{ x; -\frac{\pi}{2} < \arg x < \frac{\pi}{2} \} \)

generalized Borel summable to a solution \( y(x; C) \) analytic in

\[
S_{an} = \{ x \mid -\frac{\pi}{2} + \epsilon < \arg x < \frac{\pi}{2} - \epsilon, \; |x| > R, \; |Ce^{-x} x^\alpha| < \delta^{-1} \}
\]

**What happens to** \( y(x; C) \) **as** \( \arg x \) **approaches** \( \frac{\pi}{2} \)? (Similarly, for \(-\frac{\pi}{2}\).)
Near the boundary of the sector of analyticity

Solutions $y(x; C) \to 0$ for $x \to \infty$, $x \in d$ are analytic in $S$ for $|x|$ large.

**Question:** what happens to $y(x; C)$ as $x$ approaches $\partial S$?

**Example:** $d=1$ \[ y' + \left(1 - \frac{\alpha}{x}\right)y = g(x^{-1}, y) \quad (\lambda = 1). \]

**Formal solution:**

$$\tilde{y}(x; C) = \hat{y}_0(x) + Ce^{-x}x^\alpha \tilde{s}_1(x) + C^2 e^{-2x}x^{2\alpha} \tilde{s}_2(x) + C^3 e^{-3x}x^{3\alpha} \tilde{s}_3(x) + \ldots$$

with $\tilde{s}_k(x) = \sum_{j=0}^{\infty} \frac{y_{k,j}}{x^j}$ valid in the sector $S_{\text{trans}} = \{x; -\frac{\pi}{2} < \arg x < \frac{\pi}{2}\}$

generalized Borel summable to a solution $y(x; C)$ analytic in

$S_{\text{an}} = \{x \mid -\frac{\pi}{2} + \epsilon < \arg x < \frac{\pi}{2} - \epsilon, \ |x| > R, \ |Ce^{-x}x^\alpha| < \delta^{-1}\}$

**What happens to** $y(x; C)$ **as** $\arg x$ **approaches** $\frac{\pi}{2}$? (Similarly, for $-\frac{\pi}{2}$.)
A two scale expansion in the region with singularities

\[ \tilde{y}(x; C) = \tilde{y}_0(x) + Ce^{-x}x^\alpha \tilde{s}_1(x) + C^2 e^{-2x}x^{2\alpha} \tilde{s}_2(x) + C^3 e^{-3x}x^{3\alpha} \tilde{s}_3(x) + \ldots \]

Denote \[ Ce^{-x}x^\alpha = \xi \] For \( C \neq 0 \). Transseries:

\[ \tilde{y} = \left[ \frac{y_{0,1}}{x} + \frac{y_{0,2}}{x^2} + \ldots \right] + \xi \left[ y_{1,0} + \frac{y_{1,1}}{x} + \frac{y_{1,2}}{x^2} + \ldots \right] + \xi^2 \left[ y_{2,0} + \frac{y_{2,1}}{x} + \frac{y_{2,2}}{x^2} + \ldots \right] \]

In the region: \( x^{-k} \ll \xi \) reorder the transseries:

\[ \hat{y} = \left[ \xi y_{1,0} + \xi^2 y_{2,0} + \ldots \right] + \frac{1}{x} \left[ y_{0,1} + \xi y_{1,1} + \xi^2 y_{2,1} + \ldots \right] + \frac{1}{x^2} \left[ y_{0,2} + \xi y_{1,2} + \ldots \right] \]

with the form \( \hat{y}(x; C) = F_0(\xi) + \frac{1}{x} F_1(\xi) + \frac{1}{x^2} F_2(\xi) + \ldots \)

Note: \( F_0(0) = 0 \). Note: choose \( y_{1,0} = 1 \) (to fix \( C \)). \( \sim F'_0(0) = 1 \).

Higher dimensions - similar.
A two scale expansion in the region with singularities

\[ \tilde{y}(x; C) = \tilde{y}_0(x) + C e^{-x} x^\alpha \tilde{s}_1(x) + C^2 e^{-2x} x^{2\alpha} \tilde{s}_2(x) + C^3 e^{-3x} x^{3\alpha} \tilde{s}_3(x) + \ldots \]

Denote \( Ce^{-x} x^\alpha = \xi \) For \( C \neq 0 \). Transseries:

\[ \tilde{y} = \left[ \frac{y_{0,1}}{x} + \frac{y_{0,2}}{x^2} + \ldots \right] + \xi \left[ y_{1,0} + \frac{y_{1,1}}{x} + \frac{y_{1,2}}{x^2} + \ldots \right] + \xi^2 \left[ y_{2,0} + \frac{y_{2,1}}{x} + \frac{y_{2,2}}{x^2} + \ldots \right] + \ldots \]

In the region: \( x^{-k} \ll \xi \) reorder the transseries:

\[ \hat{y} = \left[ \xi y_{1,0} + \xi^2 y_{2,0} + \ldots \right] + \frac{1}{x} \left[ y_{0,1} + \xi y_{1,1} + \xi^2 y_{2,1} + \ldots \right] + \frac{1}{x^2} \left[ y_{0,2} + \xi y_{1,2} + \ldots \right] \]

with the form \( \hat{y}(x; C) = F_0(\xi) + \frac{1}{x} F_1(\xi) + \frac{1}{x^2} F_2(\xi) + \ldots \)

Note: \( F_0(0) = 0 \). Note: choose \( y_{1,0} = 1 \) (to fix \( C \)). \( \sim F'_0(0) = 1 \).

Higher dimensions - similar.
The series approximates solutions near singularities

Representation for $x$ near $i\mathbb{R}^+$ (recall $\lambda_1 = 1$). Denote
$$\mathcal{E}_+ = \{x; -\frac{\pi}{2} + \delta < \arg x < \frac{\pi}{2} + \delta, \Re(\lambda_jx/|x|) > c, j = 2, \ldots\}$$
$$S_{\delta_1} = \{x \in \mathcal{E}_+; |\xi(x)| < \delta_1\}$$

**Theorem** (OC, RDC, 2001)
There exists $\delta_1 > 0$ so that all $F_m$ are analytic for $|\xi| < \delta_1$ and
$$y(x) \sim F_0(\xi) + \frac{1}{x} F_1(\xi) + \frac{1}{x^2} F_2(\xi) + \ldots$$ uniformly for $x \in S_{\delta_1}, x \to \infty$.

The series is differentiable and satisfies Gevrey estimates.

It turns out that the series remains asymptotic in part of $\mathcal{E}_+ \setminus S_{\delta_1}$ near $\xi = \xi_s$ singularity of $F_0$. 
\[ y(x; C) \sim F_0(\xi) + \frac{1}{x} F_1(\xi) + \frac{1}{x^2} F_2(\xi) + \ldots \]

**The picture:** If \( \xi_s \) is an isolated singularity of \( F_0 \), calculate \( x = \tilde{x}_n \) solutions of \( \xi(x) = C_1 e^{-x} x^{\alpha_1} = \xi_s \implies \)
\[ x = \tilde{x}_n = 2n\pi i + \alpha_1 \ln(2n\pi i) + \ln C_1 - \ln \xi_s + o(1), \quad (n \to \infty) \]

Then each solution \( y(x; C) \) (specified by \( C \)) has an array of singularities at:
\[ x_n = \tilde{x}_n + o(1) = 2n\pi i + \alpha_1 \ln(2n\pi i) + \ln C_1 - \ln \xi_s + o(1), \quad (n \to \infty). \]
(almost periodic).

Moreover:
\[ y(x; C) \sim F_0(\xi(x)) + \frac{1}{x} F_1(\xi(x)) + \frac{1}{x^2} F_2(\xi(x)) + \ldots \text{ for } x \to \infty, \ x \in D_x \]
where \( D_x \) is a connected domain surrounding all \( x_n \) with \( n > N \).
(An asymptotic series valid near infinitely many singularities!)
Small neighborhoods of the poles in the array are removed.
The Painlevé equation $P_1$

\[
\frac{d^2 u}{dz^2} = 6u^2 + z
\]

**Tonquée solutions** have the same classical asymptotic expansion in the pole free sector: they differ by a constant $C$ beyond all orders.

**Plan:**
- recover the constant using transseries
- characterize the tritronquée
- find the first array of poles beyond the sector of analyticity

Consider solutions with $u(z) \sim +\sqrt{-\frac{z}{6}}$ for $z \to -\infty$.

(The family $u(z) \sim -\sqrt{-\frac{z}{6}}$ is similar.)
The Painlevé equation $P_I$

$$\frac{d^2 u}{dz^2} = 6u^2 + z$$

**Tonquée solutions** have the same classical asymptotic expansion in the pole free sector: they **differ by a constant $C$ beyond all orders**.

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(The family \( u(z) \sim -\sqrt{-\frac{z}{6}} \) is similar.)
Existence of tronquée solutions of $P_I$

**Normalization:**

\[ x = \frac{(-24z)^{5/4}}{30}; \quad u(z) = \sqrt{-z} \left( 1 - \frac{4}{25x^2} + h(x) \right) \sim \text{Boutroux form!} \]

$P_I$ normalized:

\[ h'' + \frac{1}{x} h' - h - \frac{1}{2} h^2 - \frac{392}{625} \frac{1}{x^4} = 0 \]

**Proposition**

$P_{I,\text{norm}}$ has unique $o(1)$ asy series sol as $x \to \infty$: \( \tilde{h}_0(x) = \sum_{k=4, k \text{ even}}^{\infty} \frac{c_k}{x^k} \)

The complete formal sol have the form

\[ \tilde{h}(x) = \tilde{h}_0(x) + \sum_{n \geq 1} C^n e^{-nx} \tilde{h}_n(x) \quad \text{for } \arg x \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right), \]

\[ \tilde{h}(x) = \tilde{h}_0(x) + \sum_{n \geq 1} C^n e^{nx} \tilde{h}_n(x) \quad \text{for } \arg x \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \pm i\pi \]

where \( \tilde{h}_n(x) = x^{-n/2} \tilde{s}_n(x), \quad \tilde{h}_n(x) = x^{-n/2} e^{\mp n\pi i/2} \tilde{s}_n(-x) \)

Next: Borel Summation and correspondence with actual solutions.
Existence of tonquee solutions

**Proposition** (OC, RDC, MH, 2014)
Let $h(x) = o(1)$ as $x \to \infty$ with $|\arg x| < \pi/2$.
Then $h(x) \sim \tilde{h}_0$ and there are $C_\pm$ so that, for $|x|$ large enough,

$$h(x) = \begin{cases} 
  h_0(x) + \sum_{n=1}^{\infty} C_+ e^{-nx} h_n(x) & \text{for } \arg z \in (0, \frac{\pi}{2}) \\
  h_0(x) + \sum_{n=1}^{\infty} C_- e^{-nx} h_n(x) & \text{for } \arg x \in (-\frac{\pi}{2}, 0)
\end{cases}$$

where $h_n(x) = \mathcal{L}_\phi H_n$, $H_n(p) = B \tilde{h}_n$, ($-\phi = \arg x$).
Similar statements for $h(x) = o(1)$ in $|\arg x| < \pi/2$.

**Stokes phenomenon**

$$C_+ - C_- = -\mu \text{ (there is only one free constant)}$$

$$(\mathcal{L}_0 H_0 - \mathcal{L}_0 H_0) = -\mu e^{-x} x^{-1/2}(1 + o(1)))$$

$$\Rightarrow \exists 1\text{-param fam. sol. an. for large } |x| \text{ in RHP (and sol in LHP).}$$
Tronquée solutions

Tronque sol have the form

\[ h(x) = \begin{cases} 
  h_0(x) + \sum_{n=1}^{\infty} C_n e^{-nx} h_n(x) & \text{for } \arg x \in (0, \frac{\pi}{2}) \\
  h_0(x) + \sum_{n=1}^{\infty} C_n e^{-nx} h_n(x) & \text{for } \arg x \in (-\frac{\pi}{2}, 0) 
\end{cases} \]

If \( C_+ = 0 \), then \( \exists \ AC \ h(x) = L\phi H_0 \) for \( \arg x \in \left( \frac{\pi}{2}, \pi \right) \), then through \( \mathbb{R}_- \), collecting an \( Ce^x \) (Stokes phenomenon), and further AC for \( \arg x \in \left( \pi, \frac{3\pi}{2} \right) \).

This \( h(x) \) has asymptotic power series for \( \arg x \in \left( -\frac{\pi}{2}, \frac{3\pi}{2} \right) \).

Returning to \( u(z) \): a unique solution with asymptotic power series for \( \arg z \in \left( -\frac{2\pi}{5}, \frac{6\pi}{5} \right) \): a tritonquée solution.
For \( y = (h, h') \) we have \( \lambda_{1,2} = \pm 1, \) \( \alpha_{1,2} = -1/2. \) Let \( \xi = Ce^{-x}x^{-1/2}. \)

Substitute \( h(x) \sim \sum_{k=0}^{\infty} x^{-k}H_k(\xi(x)). \) Assume \( x^{-k} \ll \xi, \forall k. \)

\[
\sim \xi^2 H_0'' + \xi H_0' = H_0 + \frac{1}{2} H_0^2
\]

with the initial condition \( H_0(\xi) = \xi + O(\xi^2) \sim H_0(\xi) = \frac{\xi}{(\xi/12 - 1)^2} \)

\( \xi_s = 12 \) is a 2\(^{nd} \) ord. pole, and (it is shown that) so are \( x_n, \) where \( x_n \) solve

\[
Ce^{-x}x^{-1/2} = 12, \quad |x| \text{ large }, \arg x \approx \pi/2
\]

The general theorem applies...
Returning to the original variables $u(z)$:

**Proposition.** (OC, RDC, MH 2015) Let $u$ be a tronquée:

$$
\frac{d^2 u}{dz^2} = 6u^2 + z \text{ such that } u(z) \sim \sqrt{-z/6} \text{ for } z \to \infty, \ \arg(z) = \pi.
$$

Let $\epsilon > 0$ and $Z = \{z \mid \arg(z) > \frac{3}{5}\pi; |\xi(z)| < \epsilon^{-1}; |\xi(z) - 12| > \epsilon\}$.

(Note: $Z$ surrounds infinitely many poles of $u$, it starts at the antistokes line $\arg(z) = \frac{3}{5}\pi$ and extends slightly beyond the next antistokes line $\arg(z) = \frac{7}{5}\pi$. )

Then $u \sim \sqrt{-z/6} \left(1 - \frac{1}{8\sqrt{6}(-z)^{5/2}} + \sum_{k=0}^{\infty} \frac{30^k H_k(\xi)}{(-24z)^{5k/4}}\right)$ \quad (z \to \infty, \ z \in Z)

The functions $H_k$ are rational, and $H_0(\xi) = \xi(\xi/12 - 1)^{-2}$.

The expansion holds uniformly in the sector $3\pi/5 < \arg(z) < 7\pi/5$ and for $\arg z \approx 7\pi/5$, (where $H_0$ becomes dominant), down to an $o(1)$ distance of the actual poles of $u$ if $z$ is large.
Tronquée solutions of $P_5$

\[
\frac{d^2 w}{dz^2} = \left( \frac{1}{2w} + \frac{1}{w - 1} \right) \left( \frac{dw}{dz} \right)^2 - \frac{1}{z} \frac{dw}{dz} + \frac{(w - 1)^2}{z^2} \left( \alpha w + \frac{\beta}{w} \right) + \frac{\gamma w}{z} + \frac{\delta w(w + 1)}{w - 1}, \quad (\delta = -\frac{1}{2})
\]

All solutions of are meromorphic in $\mathbb{C} \setminus L$, with $L$ from 0 to $\infty$.

Asymptotic series solutions: A. Parushnikova (2012). For $\alpha \beta \delta \neq 0$

\[
w = \pm \sqrt{\frac{\beta}{\delta}} z^{-1} + O(z^{-2}), \quad w = -1 + O(z^{-1}), \quad w = \pm \sqrt{-\frac{\delta}{\alpha}} z + c_0 + O(z^{-1})
\]
Transseries approach

Let $\delta = -\frac{1}{2}$. Looked at $w \sim \sqrt{-2\beta} z^{-1} \ (z \to \infty)$.

Has a unique asymptotic ps:

$$\hat{w}_0(z) = \sqrt{-2\beta} z^{-1} + \sum_{n=2}^{\infty} w_0 n z^{-n} \ (z \to \infty)$$

Exp small terms: subs $w(z) = \hat{w}_0(z) + g(z)$ assuming $g(z) \ll z^{-n}, \ \forall n$.

$P_5$: $\hat{F}_0(z) + \hat{F}_1(z) g(z) + \hat{G}(z) g'(z) + g''(z) + [\text{Quadratic in } g, g'] = 0$

- $\tilde{w}_0$ is formal solution $\implies \hat{F}_0(z) = 0$.
- Quadratic in $g, g' \ll g, g' \implies$ neglect.
- Retain only dominant powers in $\hat{F}_1(z), \hat{G}(z)$.

We remain with...
\[ g''(z) + \frac{2}{z} g'(z) - \left(1 + \frac{2\gamma + 4\sqrt{2}\sqrt{-\beta}}{z}\right) g(z) = 0 \]

WKB yields

\[ g(z) = Ce^{\pm z} z^{-Q}(1 + o(1)), \quad Q = \gamma + 2A, \quad A = \sqrt{-2\beta} \]

Choose \( e^{-z} \) to study solutions in the rhp (or \( e^{+z} \) in the lhp).

*Note*: \( z = \infty \) is a rank one sing, no normalization needed.

We can directly apply general theorems, yielding...
Sol. of $P_5$ with $w \sim \sqrt{-2\beta} z^{-1}$ as $z \to \infty$ along $d$ in RHP

We found complete formal solutions, but to apply the general summation theorems we need to bring it to a normal form first:

$$P_V : \quad \frac{d^2w}{dz^2} = \left( \frac{1}{2w} + \frac{1}{w-1} \right) \left( \frac{dw}{dz} \right)^2 - \frac{1}{z} \frac{dw}{dz}$$

$$+ \frac{(w-1)^2}{z^2} \left( \alpha w + \frac{\beta}{w} \right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}, \quad (\delta = -\frac{1}{2})$$

(not analytic at $w = 0$ and $w = O(z^{-1}) > O(z^{-2})$.)

Substitute: $w(z) = \frac{A}{z} \left( 1 - \frac{Q}{z} + u(z) \right) \quad (A = \sqrt{-2\beta}, \quad Q = 2A + \gamma)$

Now eq. for $u$ is analytic at $u = 0$ and $u(z) = O(z^{-2})$

(since $w(x) \sim \frac{A}{z} - \frac{Q}{z^2} + O(z^{-3})$).
Existence of tronquée solutions for $P_v$

**Theorem**

(i) $\exists$ complete formal solutions in RHP:

$$\hat{w}(z; C) = \hat{w}_0(z) + \sum_{n=1}^{\infty} \left( Ce^{-x} e^{-Q} \right)^n \hat{w}_n(z), \quad \hat{w}_0(z) = \sqrt{-2\beta} z^{-1} + ...$$

where $\hat{w}_n(z)$ are power series in $z^{-1}$ (divergent).

(ii) $\hat{w}_n(z)$ are Borel summable along any direction $-\phi = \arg z \in (-\frac{\pi}{2}, 0)$, to $w_n = L_{\phi} B \hat{w}_n$ and the series

$$w_0(z) + \sum_{n=1}^{\infty} \left( Ce^{-x} e^{-Q} \right)^n w_n(z)$$

converges, for $|z|$ large enough, to a solution of $P_5$. The same one for all $\phi$.

(ii') A similar statement holds for all $-\phi = \arg z \in (0, \frac{\pi}{2})$. 

Conversely,

I. Assume \( w(z) \) solves \( P_5 \) and \( w \sim \sqrt{-2\beta} z^{-1} \) as \( z \to \infty \) along \( d \) in RHP. Then there are \( C_\pm \) so that, for \( |z| \) large enough,

\[
w(z) = \begin{cases} 
    w_0(z) + \sum_{n=1}^{\infty} \left( C_+ e^{-\phi x} x^{-Q} \right)^n w_n(z) & \text{for } \arg z \in (0, \frac{\pi}{2}) \\
    w_0(z) + \sum_{n=1}^{\infty} \left( C_- e^{-\phi x} x^{-Q} \right)^n w_n(z) & \text{for } \arg z \in \left(-\frac{\pi}{2}, 0\right)
\end{cases}
\]

where \( A^2 = -2\beta, \ Q = \gamma + 2A \) and \( w_n(z) = L_\phi B_\# w_n(z), \ \phi = -\arg z \).

II. Similar statements hold in the LHP.
Let $w(z)$ solution with

$$w(z) = w_0(z) + \sum_{n=1}^{\infty} \left( C_+ e^{-z} x^{-Q} \right)^n w_n(z) \text{ for } \arg z \in (0, \frac{\pi}{2})$$

Searching for poles for large $z$ with $\arg z \approx \frac{\pi}{2}$ we let $\xi = C_+ e^{-z} x^{-Q}$, and look for solutions

$$w(z) \sim F_0(\xi) + \frac{1}{z} F_1(\xi) + \frac{1}{z^2} F_2(\xi) + \ldots \text{ assuming } z^{-k} \ll \xi, \forall k$$

We get $F_n(\xi)=$ polynomials. No poles! ???
What is going on?

The transseries solution has the form:

\[
\hat{w}(z) = \frac{w_{01}}{z} + \frac{w_{02}}{z^2} + \frac{w_{03}}{z^3} + \ldots \\
+ Ce^{-x}z^{-Q} \left( 1 + \frac{w_{11}}{z} + \frac{w_{12}}{z^2} + \frac{w_{13}}{z^3} + \ldots \right) \\
+ \left( Ce^{-x}z^{-Q} \right)^2 \left( \frac{w_{21}}{z} + \frac{w_{22}}{z^2} + \frac{w_{13}}{z^3} + \ldots \right) \\
+ \left( Ce^{-x}z^{-Q} \right)^3 \left( \frac{w_{32}}{z^2} + \frac{w_{33}}{z^3} + \frac{w_{34}}{z^4} + \ldots \right) \\
+ \left( Ce^{-x}z^{-Q} \right)^4 \left( \frac{w_{43}}{z^3} + \frac{w_{44}}{z^4} + \ldots \right)
\]

This suggests to search for an expansion using the second scale
\[\zeta = Ce^{-x}z^{-Q-2},\] and of the form

\[w(z) \sim z^2 F_0(\zeta) + zF_1(\zeta) + F_2(\zeta) + \frac{1}{z} F_3(\zeta) + \ldots \quad \text{when} \quad z^{-n} \ll \zeta\]

Thanks, Maple!
What is going on?

The transseries solution has the form:

\[ \hat{w}(z) = \frac{w_{01}}{z} + \frac{w_{02}}{z^2} + \frac{w_{03}}{z^3} + \ldots \]

\[ + Ce^{-x} x^{-Q} \left( 1 + \frac{w_{11}}{z} + \frac{w_{12}}{z^2} + \frac{w_{13}}{z^3} + \ldots \right) \]
\[ + \left( Ce^{-x} x^{-Q} \right)^2 \left( \frac{w_{21}}{z} + \frac{w_{22}}{z^2} + \frac{w_{13}}{z^3} + \ldots \right) \]
\[ + \left( Ce^{-x} x^{-Q} \right)^3 \left( \frac{w_{32}}{z^2} + \frac{w_{33}}{z^3} + \frac{w_{34}}{z^4} + \ldots \right) \]
\[ + \left( Ce^{-x} x^{-Q} \right)^4 \left( \frac{w_{43}}{z^3} + \frac{w_{44}}{z^4} + \ldots \right) \]

This suggests to search for an expansion using the second scale
\[ \zeta = Ce^{-z} z^{-Q-2}, \] and of the form

\[ w(z) \sim z^2 F_0(\zeta) + z F_1(\zeta) + F_2(\zeta) + \frac{1}{z} F_3(\zeta) + \ldots \quad \text{when } z^{-n} \ll \zeta \]

Thanks, Maple!
What is going on?

The transseries solution has the form:

\[
\hat{w}(z) = \frac{w_{01}}{z} + \frac{w_{02}}{z^2} + \frac{w_{03}}{z^3} + \ldots \\
+ Ce^{-x}x^{-Q} \left(1 + \frac{w_{11}}{z} + \frac{w_{12}}{z^2} + \frac{w_{13}}{z^3} + \ldots \right) \\
+ \left(Ce^{-x}x^{-Q}\right)^2 \left(\frac{w_{21}}{z} + \frac{w_{22}}{z^2} + \frac{w_{13}}{z^3} + \ldots \right) \\
+ \left(Ce^{-x}x^{-Q}\right)^3 \left(\frac{w_{32}}{z^2} + \frac{w_{33}}{z^3} + \frac{w_{34}}{z^4} + \ldots \right) \\
+ \left(Ce^{-x}x^{-Q}\right)^4 \left(\frac{w_{43}}{z^3} + \frac{w_{44}}{z^4} + \ldots \right)
\]

This suggests to search for an expansion using the second scale
\[\zeta = Ce^{-z}z^{-Q-2},\text{ and of the form}\]

\[w(z) \sim z^2 F_0(\zeta) + z F_1(\zeta) + F_2(\zeta) + \frac{1}{z} F_3(\zeta) + \ldots \text{ when } z^{-n} \ll \zeta\]

Thanks, Maple!
This leads to the eq. for $F_0(\zeta)$:

$$-2\beta\alpha F_0^3 + \frac{3}{2} \frac{\zeta^2 F_0'}{F_0} - \zeta F_0' - \zeta^2 F_0'' - \frac{1}{2} F_0 = 0$$

with gen. sol.

$$F_0(\zeta) = \frac{4C_1\zeta}{-16\alpha\beta - C_1^2(\zeta - C_2)^2}$$

Condition that $F_0(\zeta) = \zeta + O(\zeta^2) \Rightarrow C_2^2 = -16\beta\alpha C_1^{-2} - 4 C_1^{-1}, C_1 = \forall$

It is determined so that $F_1$ is a rational function (has no log) \(\rightsquigarrow\)

$$F_0(\zeta) = \frac{\zeta}{1 - R\zeta - N\zeta^2}$$

where $R = (A - 1 - Q)A$, $N = -\alpha\beta - 1/4 R^2$, $C_1 = 4N$
This leads to the eq. for $F_0(\zeta)$:

$$-2\beta\alpha F^3_0 + \frac{3}{2} \frac{\zeta^2 F'_0}{F_0} - \zeta F'_0 - \zeta^2 F''_0 - \frac{1}{2} F_0 = 0$$

with gen. sol.

$$F_0(\zeta) = \frac{4C_1\zeta}{-16\alpha\beta - C_1^2(\zeta - C_2)^2}$$

Condition that $F_0(\zeta) = \zeta + O(\zeta^2) \Rightarrow C_2^2 = -16\beta\alpha C_1^{-2} - 4C_1^{-1}, \ C_1 = \forall$

It is determined so that $F_1$ is a rational function (has no log) \(\Rightarrow\)

$$F_0(\zeta) = \frac{\zeta}{1 - R\zeta - N\zeta^2}$$

where $R = (A - 1 - Q)A, \ N = -\alpha\beta - 1/4 R^2, \ C_1 = 4N$
This leads to the eq. for $F_0(\zeta)$:

$$-2\beta\alpha F_0^3 + \frac{3}{2} \frac{\zeta^2 F_0' \,^2}{F_0} - \zeta F_0' - \zeta^2 F_0'' - \frac{1}{2} F_0 = 0$$

with gen. sol.

$$F_0(\zeta) = \frac{4C_1\zeta}{-16\alpha\beta - C_1^2(\zeta - C_2)^2}$$

Condition that $F_0(\zeta) = \zeta + O(\zeta^2) \Rightarrow C_2^2 = -16\beta\alpha C_1^{-2} - 4C_1^{-1}$, $C_1 = \forall$

It is determined so that $F_1$ is a rational function (has no log) \(\Rightarrow\)

$$F_0(\zeta) = \frac{\zeta}{1 - R\zeta - N\zeta^2}$$

where $R = (A - 1 - Q)A$, $N = -\alpha\beta - 1/4R^2$, $C_1 = 4N$
This leads to the eq. for $F_0(\zeta)$:

$$-2\beta\alpha F_0^3 + \frac{3}{2} \frac{\zeta^2 F_0'}{F_0} - \zeta F_0' - \zeta^2 F_0'' - \frac{1}{2} F_0 = 0$$

with gen. sol.

$$F_0(\zeta) = \frac{4 C_1 \zeta}{-16\alpha\beta - C_1^2 (\zeta - C_2)^2}$$

Condition that $F_0(\zeta) = \zeta + O(\zeta^2) \Rightarrow C_2^2 = -16\beta\alpha C_1^{-2} - 4 C_1^{-1}, \ C_1 = \forall$

It is determined so that $F_1$ is a rational function (has no log) \(\rightsquigarrow\)

$$F_0(\zeta) = \frac{\zeta}{1 - R\zeta - N\zeta^2}$$

where $R = (A - 1 - Q)A$, $N = -\alpha\beta - 1/4 R^2$, $C_1 = 4N$. 
Induction: $F_n$ with $n \geq 1$ satisfy the linear non-homogeneous equations

$$
\xi^2 F''_n + \xi \left(1 - 3\xi \frac{F'}{F}\right) F'_n + \left(\frac{3}{2} \frac{\xi^2 (F')^2}{F^2} + \frac{1}{2} - 3 A^2 aF^2\right) F_n = R_n (\xi, F(\xi), F_{-1}(\xi), \ldots, F_{n-1}(\xi))
$$

where $R_n$ are rational functions.

**Proposition**

There exist constants $C_1, C_2$ in $F_{n-1}$ so that $F(\xi) = \xi + O(\xi^2)$ ($\xi \to 0$) and so that all solutions $F_n(\xi)$ are analytic at $\xi = 0$.

As a consequence, the only singularities of $F_n$ are poles, when $N\xi^2 + R\xi - 1 = 0$, for all $n \geq 0$.

For example

$$
F_1 (\zeta) = \frac{\zeta (N\zeta^2 + 1)}{(N\zeta^2 + R\zeta - 1)^2} \left[ -\frac{(2 N\zeta + R)}{2 (N\zeta^2 + 1) N} C_3 + C_4 \right]
$$

Next: a contractive argument is used to show that there is a solution asymptotic to this expansion.
Solve for $z$:
\[ N\zeta^2 + R\zeta - 1 = 0, \quad \zeta = Ce^{-z}z^{-Q-2}, \quad |z| \text{ large}, \quad \arg z \approx \pi/2. \]

**Further questions:**

- Numerical verification would be great.
- For $C_+ = 0$ there exists tritonquee solutions, as explained for $P_I$. Where are their first array of poles located?
- How do the other two families of tronquee solutions behave?
- Does $P_\nu$ admit entire solutions (for special parameters)?
Thank You!