Tronquée Solutions of Painlevé equations

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Painlevé equations and Applications. Workshop in memory of A.A. Kapaev

Generic solutions of the Painlevé equations have singularities (poles) in any sector towards ∞ in the complex plane. But some are free of poles for large z in sectors:

$P_I: \ u'' = 6u^2 - z$ *Boutroux:*

- there is a one-parameter family with no poles at infinity in a two adjacent sectors bd, by $\arg z = \frac{2k\pi i}{5}$, *tronquée solutions*.
- There are solutions with no poles in four such sectors: *tritronquée*

Dubrovin's conjecture (2009): tonquée have no poles in the sector.
Proved so by O. Costin, H. Huang, S. Tanveer (2015)

$P_{II}: \ u'' = zu + 2u^3 + \alpha$ *Boutroux:* sectors bd. by $\arg z = \frac{2k\pi i}{6}$
Existence of tronquée solutions proved by N. Joshi and M. Mazzocco (2003)

Novokshenov's conjecture (2014): tonquée of any P_n have no poles in the sector.
Proved so for the Hastings-McLeod solution (2-tronqée) by M. Huang, S-X Xu, Lun Zhang (2015).

P_{III}, P_{IV}

- Existence of tronquée solutions: Lin, Dai, Tibboel (2013)
- Existence of tronquée & position of the first array of poles, X. Xia (submitted)

P_V

Andreev, Kitaev - the exp small correction (1997)
S. Shimomura (at $w = 1$) (2011)

A. Kapaev's work on tritronquée solutions, including he calculated the exp small term for P_I

Methods: classical analysis, isomonodromy methods.

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Our approach (RDC, OC, MH, S. Tanveer, Xiaoyue Xia):

We are studying truncated solutions using generalized Borel summation of their transseries (multi-instanton expansions).

Advantages:

- Once the equation is normalized ($\exists$ almost algorithmic procedures)
- ...there are general theorems establishing

$$\{\text{transseries solutions, in sectors}\} \longleftrightarrow \{\text{actual solutions an. there to } \infty\}$$

- The location of the first array of poles, beyond the sector of analyticity is also obtained.

New results with these techniques

- Combined with novel methods in the finite plane, Dubrovin's conjecture was proved.
- New (readable!) proof for P_1 is coming soon. (OC, RDC)
- Combined with novel methods in the finite plane, Novokshenko's conjecture was proved for McLeod solution of P_2.
- Obtained the Stokes multiplier directly for P_1. (OC, RDC, MH, 2015)
- Location of first arrays of poles for P_3,P_4 (X.X.)

NEW:

- Combined with *a new type of convergent expansions* $\mapsto$ very efficient numerical methods to calculate solutions of P_n. (O Costin, G. Dunne)
- I will present some
  - recent work on P_I (OC, RDC)
  - ongoing work on P_5 (RDC)

Example: linear equation $\boxed{y' + y = x^{-2}}$
Point $x = \infty$ is a *rank 1 irregular sing. pt.*

Has unique power series sol as $x \to \infty$: $\tilde{y}_0(x) = \sum_{n=2}^{\infty} \frac{(n-1)!}{x^n}$. Divergent.

General solution:

$$y(x; C) = y_0(x) + Ce^{-x}, \;\; y_0(x) = e^{-x} \int^x e^s s^{-2} ds \sim \tilde{y}_0(x) \qquad (x \to \infty)$$

Phenomena at irreg. sing. pts. :

- power series sol are divergent
- loss of information (1-param. fam. of sol. asy. to the same series)
- asymptoticity holds only in sectors (here for $|\arg x| < \pi/2$)

Natural: complete formal solution $\tilde{y}(x) = \tilde{y}_0(x) + Ce^{-x}$ for $x \to +\infty$.

Intro to transseries expansions (multi-instantons)

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Eq. $y' + y = \dfrac{1}{x^2}$ has
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Simplest example of transseries.

Features:

- It satisfies the equation where it originated.
- It is not an asymptotic expansion à la Poincaré ($Ce^{-x} \ll x^{-n}\ \forall n$),
- but it is well ordered for $|\arg x| < \pi/2$ (terms are decreasing).
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Nonlinear example: $$y' + y = \frac{1}{x^2} + y^4$$

Unique power series sol: $\tilde{y}_0(x) = \frac{1}{x^2} + \frac{2}{x^3} + \frac{6}{x^4} + \dots \ (x \to \infty)$

Divergent.

Smaller terms? Substitute $y = \tilde{y}_0(x) + g(x)$ assuming $g(x) \ll x^{-n}\ \forall n$.

Obtain: $\tilde{\mathcal{F}}_0(z) + \tilde{\mathcal{F}}_1(z)g(z) + g'(z) + [\text{Quadratic in } g] = 0$

- $\tilde{y}_0(x)$ is formal solution $\implies \tilde{\mathcal{F}}_0(z) = 0$.
- Quadratic $\ll g \implies$ neglect.

$\rightsquigarrow g' + g \sim 4\tilde{y}_0^3 g \quad \implies \quad g(x) \sim Ce^{-x}\tilde{y}_1(x)$.

Only $\tilde{y}_0(x) + Ce^{-x}\tilde{y}_1(x)$ is not an exact sol. Look for even smaller terms.

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A complete formal solution:

$$\tilde{y}(x) = \tilde{y}_0(x) + Ce^{-x}\tilde{y}_1(x) + C^2e^{-2x}\tilde{y}_2(x) + C^3e^{-3x}\tilde{y}_3(x) + \dots$$

where $\tilde{y}_n(x)$ are divergent power series and C is a free parameter.

General 1-dim

$$\boxed{y' + \left(\lambda - \frac{\alpha}{x}\right) y = g(x^{-1}, y)}, \quad g = O(x^{-2}) + O(y^2) \ (x \to \infty, y \to 0)$$

has a complete formal solution (in the RHP)

$$\tilde{y}(x) = \tilde{y}_0(x) + Ce^{-\lambda x}\tilde{y}_1(x) + C^2e^{-2\lambda x}\tilde{y}_2(x) + C^3e^{-3\lambda x}\tilde{y}_3(x) + \dots$$

where $\tilde{y}_n(x) = x^{n\alpha}\tilde{s}_n(x)$, where $\tilde{s}_n(x)$ are integer power series.
Valid for $x \to \infty$ with $|\arg(\lambda x)| < \pi/2$.

Systems with $x = \infty$ a *rank 1 irregular singularity*:

Normal form:

$$\mathbf{y}' + \left(\Lambda - \frac{1}{x}A\right)\mathbf{y} = \mathbf{g}(x^{-1}, \mathbf{y})$$

with $\mathbf{g}$ analytic at $(0, \mathbf{0})$, with $\mathbf{g}(x^{-1}, \mathbf{y}) = O(x^{-2}) + O(|\mathbf{y}|^2)$

If Λ, A are diagonalizable, then

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots \lambda_d), \; A = \mathrm{diag}(\alpha_1, \ldots \alpha_d)$$

(nonresonant) have formal solution:

$$\tilde{\mathbf{y}} = \tilde{\mathbf{y}}(x; \mathbf{C}) = \tilde{\mathbf{y}}_0(x) + \sum_{\mathbf{k} \in \mathbb{N}^d \setminus \mathbf{0}} \mathbf{C}^{\mathbf{k}} e^{-\boldsymbol{\lambda} \cdot \mathbf{k} x} \tilde{\mathbf{y}}_{\mathbf{k}}(x)$$

where $\tilde{\mathbf{y}}_{\mathbf{k}}(x) = x^{\boldsymbol{\alpha} \cdot \mathbf{k}} \tilde{\mathbf{s}}_{\mathbf{k}}(x)$ are power series (divergent), determined algorithmically.
(If resonant - also logs.)

Note: the free parameters are beyond all orders of the unique power series.

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Systems with ∞ a rank 1 singularity $\mathbf{y}' + \left(\Lambda - \frac{1}{x}A\right)\mathbf{y} = \mathbf{g}(x^{-1}, \mathbf{y})$ have formal solutions:

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Introduced by Fabris (1885). Studied by Cope (1934).

Vastly generalized by Ecalle (1981) to formal expressions closed under all operations.

In logic. More recently, in physics (QFT, in particular).

So: solutions $\mathbf{y} \to 0$ as $x \to \infty$ have a unique asymptotic power series:

$$\mathbf{y} \sim \tilde{\mathbf{y}}_{\mathbf{0}} \equiv \sum_{n}^{+\infty} x^{-n}\,\mathbf{y}_{0,n} \qquad \text{(usually divergent)}$$

Conversely:

Theorem [Wasow]

If $\tilde{\mathbf{y}}_{\mathbf{0}}$ formally solves $\mathbf{y}' + \left(\Lambda - \frac{1}{x}A\right)\mathbf{y} = \mathbf{g}(x^{-1}, \mathbf{y})$ (and a nonres. cond. on λ_j, α_j) then there exists a true solution

$$\mathbf{y} \sim \hat{\mathbf{y}}_{\mathbf{0}}, \quad |x| > R,\ x \in \mathcal{S} = \text{some sector opening } < \pi$$

Usually, **there are many** such solutions.

Recall that the free parameters are beyond all orders of $\tilde{\mathbf{y}}_{\mathbf{0}}$...

Correspondence between formal and actual solutions: Generalized Borel Summation (O Costin 1995, '98)

Example: linear equation $\boxed{y' + y = x^{-2}}$

Formal solution $\tilde{y}(x; C) = \sum_{n \geqslant 2} (n-1)!\, x^{-n} + C\mathrm{e}^{-x}$

Since $\mathcal{L}(p^{n-1}) = (n-1)! x^{-n}$ it is natural to attempt $\tilde{y}_0 = \mathcal{L}(\mathit{function})$.

Recall: Borel transform is formal $\mathcal{L}^{-1}$: $\mathcal{B}(x^{-\alpha}) = p^{\alpha-1}/\Gamma(\alpha)$ for $\alpha > 0$.

Let $\boxed{y(x) = \mathcal{L}_\phi Y(x) := \int_{e^{i\phi}\mathbb{R}_+} e^{-px} Y(p) dp}$ eq. becomes $Y(p) = \dfrac{p}{1-p}$.

We can integrate for $\phi \neq 0$, obtaining:

$$y_0^+(x) = \mathcal{L}_\phi Y(x) \ \text{ for } -\phi = \arg x \in \left(0, \frac{\pi}{2}\right)$$

and

$$y_0^-(x) = \mathcal{L}_\phi Y(x) \ \text{ for } -\phi = \arg x \in \left(-\frac{\pi}{2}, 0\right)$$

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Correspondence between formal and actual solutions: Generalized Borel Summation (O Costin 1995, '98)

Example: linear equation $\boxed{y' + y = x^{-2}}$

Formal solution $\tilde{y}(x; C) = \sum_{n \geqslant 2} (n-1)!\, x^{-n} + C\mathrm{e}^{-x}$

Since $\mathcal{L}(p^{n-1}) = (n-1)! x^{-n}$ it is natural to attempt $\tilde{y}_0 = \mathcal{L}(\textit{function})$.

Recall: Borel transform is formal $\mathcal{L}^{-1}$: $\mathcal{B}(x^{-\alpha}) = p^{\alpha-1}/\Gamma(\alpha)$ for $\alpha > 0$.

Let $\boxed{y(x) = \mathcal{L}_\phi Y(x) := \int_{e^{i\phi}\mathbb{R}_+} e^{-px} Y(p) dp}$ eq. becomes $Y(p) = \dfrac{p}{1-p}$.

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- does not depend on ϕ
- can be analytically continued in the RHP (and beyond)

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Take $\mathcal{L}^{-1} \implies (1-p)Y(p) = 1 + Y^{*4}(p)$

- Clearly $\exists!$ solution $Y_0(p)$ analytic at $p=0$. It is analytic for $|p|<1$.
- Clearly $Y_0(p)$ is singular at $p=1$.
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General result

Theorem (O. Costin, 1998)

Let $\boxed{\mathbf{y}' + \left(\Lambda - \frac{1}{x}A\right)\mathbf{y} = \mathbf{g}(x^{-1}, \mathbf{y})}$ nonres., Λ, A diag., $\mathbf{g}$ analytic at $(0,0)$.

Any formal solution $\tilde{\mathbf{y}} = \tilde{\mathbf{y}}(x; \mathbf{C}) = \tilde{\mathbf{y}}_0(x) + \sum_{\mathbf{k}\in\mathbb{N}^d\setminus\mathbf{0}} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} \tilde{\mathbf{y}}_{\mathbf{k}}(x)$

can be summed along any direction not an antistokes lines $(\pm i\overline{\lambda}_j\mathbb{R}_+)$

The series $\mathbf{y} = \mathbf{y}_0(x) + \sum_{\mathbf{k}\in\mathbb{N}^d\setminus\mathbf{0}} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} \mathbf{y}_{\mathbf{k}}(x)$ converges for large $|x|$

to an actual solution

analytic on open sector bounded by two consecutive antistokes lines.

Conversely, any such solutions is a summation of a transseries.

Also: RESURGENCE all the $\mathbf{y}_n$ can be recovered from $\mathbf{y}_0$!

$$\mathbf{y}_n(x) = \sum_j \alpha_{j,n} \int_{d_{j,n}} \mathrm{e}^{-px}\, \mathbf{Y}_0(p)\, dp$$

balanced averages of Laplace transforms along paths winding in prescribed ways among $p = k\lambda j$.

(1957-59) Iwano showed that $\mathbf{y}(x;\mathbf{C}) = \mathbf{y}_0(x) + \sum \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}} \mathbf{y}_{\mathbf{k}}(x)$ with $\mathbf{y}_{\mathbf{k}}(x)$ analytic, and convergent in sectors.
(1981) Ecalle constructed the summation of transseries (formal solutions of most problems), establishing an isomorphism with a class of functions ("analyzable").
(1990) Balser, Braaksma, Ramis, Sibuya proved multisummability of formal power series solutions of linear equations.
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(1998) O. Costin proved generalized Borel summation for transseries solutions of rank 1, their 1-to-1 correspondence with solutions $y(x) \to 0$ (in a sector), and compatibility with all operations.
(2001-04) Braaksma proved similar results for solutions of difference equations.

Near the boundary of the sector of analyticity

Solutions $\mathbf{y}(x;C) \to 0$ for $x \to \infty,\ x \in d$ are analytic in S for $|x|$ large.
Question: what happens to $\mathbf{y}(x;C)$ as x approaches ∂S?

Example: d=1 $\qquad y' + \left(1 - \frac{\alpha}{x}\right) y = g(x^{-1}, y) \qquad (\lambda = 1).$

Formal solution:

$$\tilde{y}(x;C) = \hat{y}_0(x) + Ce^{-x}x^{\alpha}\tilde{s}_1(x) + C^2e^{-2x}x^{2\alpha}\tilde{s}_2(x) + C^3e^{-3x}x^{3\alpha}\tilde{s}_3(x) + \dots$$

with $\tilde{s}_k(x) = \sum_{j=0}^{\infty} \frac{y_{k,j}}{x^j}$ valid in the sector $S_{trans} = \{x; -\frac{\pi}{2} < \arg x < \frac{\pi}{2}\}$

generalized Borel summable to a solution $y(x;C)$ analytic in
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What happens to $y(x;C)$ **as** $\arg x$ **approaches** $\frac{\pi}{2}$**?** (Similarly, for $-\frac{\pi}{2}$.)

A two scale expansion in the region with singularities

$$\tilde{y}(x;C) = \tilde{y}_0(x) + Ce^{-x}x^{\alpha}\tilde{s}_1(x) + C^2e^{-2x}x^{2\alpha}\tilde{s}_2(x) + C^3e^{-3x}x^{3\alpha}\tilde{s}_3(x) + \dots$$

Denote $\boxed{Ce^{-x}x^{\alpha} = \xi}$ For $C \neq 0$!. Transseries:

$$\tilde{y} = \left[\frac{y_{0,1}}{x} + \frac{y_{0,2}}{x^2} + \dots\right] + \xi\left[y_{1,0} + \frac{y_{1,1}}{x} + \frac{y_{1,2}}{x^2} + \dots\right] + \xi^2\left[y_{2,0} + \frac{y_{2,1}}{x} + \frac{y_{2,2}}{x^2}\right.$$

In the region: $x^{-k} \ll \xi$ reorder the transseries:

$$\hat{y} = \left[\xi y_{1,0} + \xi^2 y_{2,0} + \dots\right] + \frac{1}{x}\left[y_{0,1} + \xi y_{1,1} + \xi^2 y_{2,1} + \dots\right] + \frac{1}{x^2}\left[y_{0,2} + \xi y_{1,2}\right.$$

with the form $\hat{y}(x;C) = F_0(\xi) + \frac{1}{x}F_1(\xi) + \frac{1}{x^2}F_2(\xi) + \dots$

Note: $F_0(0) = 0$. *Note:* choose $y_{1,0} = 1$ (to fix C). $\rightsquigarrow$ $F_0'(0) = 1$.

Higher dimensions - similar.

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The series approximates solutions near singularities

Representation for x near $i\mathbb{R}_+$ (recall $\lambda_1 = 1$). Denote

$\mathcal{E}_+ = \{x\,;\, -\frac{\pi}{2} + \delta < \arg x < \frac{\pi}{2} + \delta,\ \Re(\lambda_j x/|x|) > c,\ j = 2, \ldots .\}$

$\mathcal{S}_{\delta_1} = \{x \in \mathcal{E}_+\,;\, |\xi(x)| < \delta_1\}$

Theorem (OC, RDC, 2001)

There exists $\delta_1 > 0$ so that all $\mathbf{F}_m$ are analytic for $|\xi| < \delta_1$ and

$\mathbf{y}(x) \sim \mathbf{F}_0(\xi) + \frac{1}{x}\mathbf{F}_1(\xi) + \frac{1}{x^2}\mathbf{F}_2(\xi) + \ldots$ uniformly for $x \in \mathcal{S}_{\delta_1}$, $x \to \infty$.

The series is differentiable and satisfies Gevrey estimates.

It turns out that **the series remains asymptotic** in part of $\mathcal{E}_+ \setminus \mathcal{S}_{\delta_1}$ near $\xi = \xi_s$ **singularity** of $\mathbf{F}_0$.

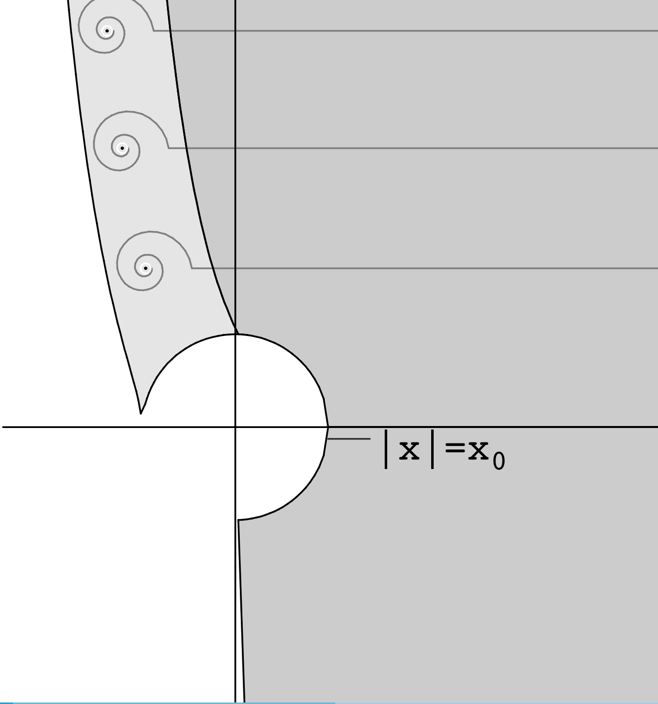
$$\mathbf{y}(x;\mathbf{C}) \sim \mathbf{F}_0(\xi) + \frac{1}{x}\mathbf{F}_1(\xi) + \frac{1}{x^2}\mathbf{F}_2(\xi) + \dots$$

The picture: If ξ_s is an isolated singularity of $\mathbf{F}_0$, calculate $x = \tilde{x}_n$ solutions of $\xi(x) = C_1 e^{-x} x^{\alpha_1} = \xi_s \implies$
$x = \tilde{x}_n = 2n\pi i + \alpha_1 \ln(2n\pi i) + \ln C_1 - \ln \xi_s + o(1), \quad (n \to \infty)$

Then each solution $\mathbf{y}(x;\mathbf{C})$ (specified by $\mathbf{C}$) has an array of singularities at:

$x_n = \tilde{x}_n + o(1) = 2n\pi i + \alpha_1 \ln(2n\pi i) + \ln C_1 - \ln \xi_s + o(1), \quad (n \to \infty)$.
(almost periodic).

Moreover:
$\mathbf{y}(x;C) \sim \mathbf{F}_0(\xi(x)) + \frac{1}{x}\mathbf{F}_1(\xi(x)) + \frac{1}{x^2}\mathbf{F}_2(\xi(x)) + \dots$ for $x \to \infty,\ x \in D_x$
where D_x is a connected domain surrounding all x_n with $n > N$.
(An asymptotic series valid near infinitely many singularities!)


|x|=x0

The Painlevé equation P_{I}

$$\frac{d^2u}{dz^2} = 6u^2 + z$$

Tonquée solutions have the same classical asymptotic expansion in the pole free sector: they differ by a constant *C beyond all orders*.

Plan:

- recover the constant using transseries
- characterize the tritronquée
- find the first array of poles beyond the sector of analyticity

Consider solutions with $u(z) \sim +\sqrt{\frac{-z}{6}}$ for $z \to -\infty$.

(The family $u(z) \sim -\sqrt{\frac{-z}{6}}$ is similar.)

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Existence of tronquée solutions of P_I

Normalization:

$$x = \frac{(-24z)^{5/4}}{30};\ \ u(z) = \sqrt{\frac{-z}{6}}\left(1 - \frac{4}{25x^2} + h(x)\right) \rightsquigarrow \text{Boutroux form!}$$

P_I normalized: $h'' + \frac{1}{x}h' - h - \frac{1}{2}h^2 - \frac{392}{625}\frac{1}{x^4} = 0$

Proposition

$\mathrm{P}_{I,norm}$ has unique $o(1)$ asy series sol as $x \to \infty$: $\tilde{h}_0(x) = \sum_{k=4,k\ \ even}^{\infty} \frac{c_k}{x^k}$

The complete formal sol have the form

$$\tilde{h}(x) = \tilde{h}_0(x) + \sum_{n\geqslant 1} C^n e^{-nx}\tilde{h}_n(x) \quad \text{for } \arg x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

$$\tilde{h}(x) = \tilde{h}_0(x) + \sum_{n\geqslant 1} C^n e^{nx}\tilde{\tilde{h}}_n(x) \quad \text{for } \arg x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \pm i\pi$$

where $\tilde{h}_n(x) = x^{-n/2}\tilde{s}_n(x)$, $\tilde{\tilde{h}}_n(x) = x^{-n/2}e^{\mp n\pi i/2}\tilde{s}_n(-x)$

Next: Borel Summation and correspondence with actual solutions

Proposition (OC, RDC, MH, 2014)
Let $h(x) = o(1)$ as $x \to \infty$ with $|\arg x| < \pi/2$.
Then $h(x) \sim \tilde{h}_0$ and there are $C_\pm$ so that, for $|x|$ large enough,

$$h(x) = \begin{cases} h_0(x) + \sum_{n=1}^{\infty} C_+^n e^{-nx} h_n(x) & \text{for } \arg z \in (0, \frac{\pi}{2}) \\ h_0(x) + \sum_{n=1}^{\infty} C_-^n e^{-nx} h_n(x) & \text{for } \arg x \in (-\frac{\pi}{2}, 0) \end{cases}$$

where $h_n(x) = \mathcal{L}_\phi H_n$, $H_n(p) = \mathcal{B}\tilde{h}_n$, $(-\phi = \arg x)$.
Similar statements for $h(x) = o(1)$ in $|\arg x| < \pi/2$.

Stokes phenomenon $C_+ - C_- = -\mu$ (there is only one free constant)
$(\mathcal{L}_{0-} H_0 - \mathcal{L}_{0+} H_0 = -\mu e^{-x} x^{-1/2}(1 + o(1)))$

$\implies \exists$ 1-param fam. sol. an. for large $|x|$ in RHP (and sol in LHP).

Tritonquée solutions

Tronquee sol have the form

$$h(x) = \begin{cases} h_0(x) + \sum_{n=1}^{\infty} C_+^n e^{-nx} h_n(x) & \text{for } \arg z \in (0, \frac{\pi}{2}) \\ h_0(x) + \sum_{n=1}^{\infty} C_-^n e^{-nx} h_n(x) & \text{for } \arg x \in (-\frac{\pi}{2}, 0) \end{cases}$$

If $C_+ = 0$, then $\exists$ AC $h(x) = \mathcal{L}_\phi H_0$ for $\arg x \in \left(\frac{\pi}{2}, \pi\right)$,
then through $\mathbb{R}_-$, collecting an Ce^x (Stokes phenomenon),
and further AC for $\arg x \in \left(\pi, \frac{3\pi}{2}\right)$.

This $h(x)$ has asymptotic power series for $\arg x \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

Returning to $u(z)$: a unique solution with asymptotic power series for $\arg z \in \left(-\frac{2\pi}{5}, \frac{6\pi}{5}\right)$: a tritonquee solution.

First array of poles beyond the sector of analyticity

For $\mathbf{y} = (h, h')$ we have $\lambda_{1,2} = \pm 1,\ \alpha_{1,2} = -1/2$. Let $\xi = Ce^{-x}x^{-1/2}$.

Substitute $h(x) \sim \sum_{k=0}^{\infty} x^{-k} H_k(\xi(x))$. Assume $x^{-k} \ll \xi,\ \forall k$.

$\rightsquigarrow \quad \xi^2 H_0'' + \xi H_0' = H_0 + \frac{1}{2} H_0^2$

with the initial condition $H_0(\xi) = \xi + O(\xi^2)$ $\rightsquigarrow$ $H_0(\xi) = \dfrac{\xi}{(\xi/12 - 1)^2}$

$\xi_s = 12$ is a 2^{nd} ord. pole, and (it is shown that) so are x_n, where x_n solve

$$Ce^{-x}x^{-1/2} = 12, \qquad |x| \text{ large}, \arg x \approx \pi/2$$

The general theorem applies...

Returning to the original variables $u(z)$:

Proposition. (OC, RDC, MH 2015) Let u be a tronquée:

$\dfrac{d^2u}{dz^2} = 6u^2 + z$ such that $u(z) \sim \sqrt{-z/6}$ for $z \to \infty,\ \arg(z) = \pi$.

Let $\epsilon > 0$ and $\mathcal{Z} = \{z \,|\, \arg(z) > \frac{3}{5}\pi; |\xi(z)| < \epsilon^{-1};\ |\xi(z) - 12| > \epsilon\}$.

(Note: $\mathcal{Z}$ surrounds infinitely many poles of u, it starts at the antistokes line $\arg(z) = \frac{3}{5}\pi$ and extends slightly beyond the next antistokes line $\arg(z) = \frac{7}{5}\pi$.)

$$\text{Then } u \sim \sqrt{\frac{-z}{6}}\left(1 - \frac{1}{8\sqrt{6}(-z)^{5/2}} + \sum_{k=0}^{\infty} \frac{30^k H_k(\xi)}{(-24z)^{5k/4}}\right) \quad (z \to \infty,\ z \in \mathcal{Z})$$

The functions H_k are rational, and $H_0(\xi) = \xi(\xi/12 - 1)^{-2}$.

The expansion holds uniformly in the sector $3\pi/5 < \arg(z) < 7\pi/5$ and for $\arg z \approx 7\pi/5$, (where H_0 becomes dominant), down to an $o(1)$ distance of the actual poles of u if z is large.

Tronquée solutions of P_5

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}, \quad (\delta = -\frac{1}{2})$$

All solutions of are meromorphic in $\mathbb{C} \setminus L$, with L from 0 to ∞.

Asymptotic series solutions: A. Parushnikova (2012). For $\alpha\beta\delta \neq 0$

$$w = \pm\sqrt{\frac{\beta}{\delta}} z^{-1} + O(z^{-2}), \;\; w = -1 + O(z^{-1}), \;\; w = \pm\sqrt{\frac{-\delta}{\alpha}} z + c_0 + O(z^{-1})$$

Transseries approach

Let $\delta = -\frac{1}{2}$. Looked at $w \sim \sqrt{-2\beta}\, z^{-1}$ $(z \to \infty)$.

Has a unique asymptotic ps:
$\hat{w}_0(z) = \sqrt{-2\beta}\, z^{-1} + \sum_{n=2}^{\infty} w_{0n} z^{-n}$ $(z \to \infty)$

Exp small terms: subs $w(z) = \hat{w}_0(z) + g(z)$ assuming $g(z) \ll z^{-n}$, $\forall n$.

P_5: $\hat{\mathcal{F}}_0(z) + \hat{\mathcal{F}}_1(z) g(z) + \hat{\mathcal{G}}(z) g'(z) + g''(z) + [\text{Quadratic in } g, g'] = 0$

- $\tilde{w}_0$ is formal solution $\implies$ $\hat{\mathcal{F}}_0(z) = 0$.
- Quadratic in $g, g' \ll g,\ g'$ $\implies$ neglect.
- Retain only dominant powers in $\hat{\mathcal{F}}_1(z)$, $\hat{\mathcal{G}}(z)$.

We remain with...

$$g''(z) + \frac{2}{z} g'(z) - \left(1 + \frac{2\gamma + 4\sqrt{2}\sqrt{-\beta}}{z}\right) g(z) = 0$$

WKB yields

$$g(z) = Ce^{\pm z} z^{-Q}(1 + o(1)), \quad Q = \gamma + 2A, \;\; A = \sqrt{-2\beta}$$

Choose e^{-z} to study solutions in the rhp (or e^{+z} in the lhp).

Note: $z = \infty$ is a rank one sing, no normalization needed.

We can directly apply general theorems, yielding...

Sol. of P_5 with $w \sim \sqrt{-2\beta}\, z^{-1}$ as $z \to \infty$ along d in RHP

We found complete formal solutions, but to apply the general summation theorems we need to bring it to a normal from first:

$$\mathrm{P_V}: \;\; \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}, \quad (\delta = -\frac{1}{2})$$

(not analytic at $w = 0$ and $w = O(z^{-1}) > O(z^{-2})$.)

Substitute: $w(z) = \frac{A}{z}\left(1 - \frac{Q}{z} + u(z)\right)$ $\quad (A = \sqrt{-2\beta},\; Q = 2A + \gamma)$

Now eq. for u is analytic at $u = 0$ and $u(z) = O(z^{-2})$
(since $w(x) \sim \frac{A}{z} - \frac{Q}{z^2} + O(z^{-3})$).

Theorem

(i) $\exists$ complete formal solutions in RHP:

$$\hat{w}(z;C) = \hat{w}_0(z) + \sum_{n=1}^{\infty} \left(Ce^{-x}x^{-Q}\right)^n \hat{w}_n(z), \quad \hat{w}_0(z) = \sqrt{-2\beta}\, z^{-1} + \ldots$$

where $\hat{w}_n(z)$ are power series in z^{-1} (divergent).

(ii) $\hat{w}_n(z)$ are Borel summable along any direction $-\phi = \arg z \in (-\frac{\pi}{2}, 0)$, to $w_n = \mathcal{L}_\phi \mathcal{B} \hat{w}_n$ and the series

$$w_0(z) + \sum_{n=1}^{\infty} \left(Ce^{-x}x^{-Q}\right)^n w_n(z)$$

converges, for $|z|$ large enough, to a solution of P_5. The same one for all ϕ.

(ii') A similar statement holds for all $-\phi = \arg z \in (0, \frac{\pi}{2})$.

Conversely,

I. Assume $w(z)$ solves P_5 and $w \sim \sqrt{-2\beta}\, z^{-1}$ as $z \to \infty$ along d in RHP. Then there are $C_\pm$ so that, for $|z|$ large enough,

$$w(z) = \begin{cases} w_0(z) + \sum_{n=1}^{\infty} \left(C_+ e^{-x} x^{-Q}\right)^n w_n(z) & \text{for } \arg z \in (0, \frac{\pi}{2}) \\ w_0(z) + \sum_{n=1}^{\infty} \left(C_- e^{-x} x^{-Q}\right)^n w_n(z) & \text{for } \arg z \in (-\frac{\pi}{2}, 0) \end{cases}$$

where $A^2 = -2\beta$, $Q = \gamma + 2A$ and $w_n(z) = \mathcal{L}_\phi \mathcal{B} \hat{w}_n(z)$, $\phi = -\arg z$.

II. Similar statements hold in the LHP.

Array of poles near regular sectors of tronquée solutions

Let $w(z)$ solution with

$$w(z) = w_0(z) + \sum_{n=1}^{\infty} \left(C_+ e^{-x} x^{-Q} \right)^n w_n(z) \text{ for } \arg z \in (0, \frac{\pi}{2})$$

Searching for poles for large z with $\arg z \approx \frac{\pi}{2}$ we let $\xi = C_+ e^{-x} x^{-Q}$, and look for solutions

$$w(z) \sim F_0(\xi) + \frac{1}{z} F_1(\xi) + \frac{1}{z^2} F_2(\xi) + \dots \quad \text{assuming } z^{-k} \ll \xi, \forall k$$

We get $F_n(\xi)$=polynomials. No poles! ???

What is going on?

The transseries solution has the form:

$$\hat{w}(z) = \frac{w_{01}}{z} + \frac{w_{02}}{z^2} + \frac{w_{03}}{z^3} + \dots$$
$$+ Ce^{-x}x^{-Q}\left(1 + \frac{w_{11}}{z} + \frac{w_{12}}{z^2} + \frac{w_{13}}{z^3} + \dots\right)$$
$$+ \left(Ce^{-x}x^{-Q}\right)^2\left(\frac{w_{21}}{z} + \frac{w_{22}}{z^2} + \frac{w_{13}}{z^3} + \dots\right)$$
$$+ \left(Ce^{-x}x^{-Q}\right)^3\left(\frac{w_{32}}{z^2} + \frac{w_{33}}{z^3} + \frac{w_{34}}{z^4} + \dots\right)$$
$$+ \left(Ce^{-x}x^{-Q}\right)^4\left(\frac{w_{43}}{z^3} + \frac{w_{44}}{z^4} + \dots\right)$$

This suggests to search for an expansion using the second scale $\zeta = Ce^{-z}z^{-Q-2}$, and of the form

$$w(z) \sim z^2 F_0(\zeta) + zF_1(\zeta) + F_2(\zeta) + \frac{1}{z}F_3(\zeta) + \dots \quad \text{when } z^{-n} \ll \zeta$$

Thanks, Maple!

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This suggests to search for an expansion using the second scale $\zeta = Ce^{-z}z^{-Q-2}$, and of the form

$$w(z) \sim z^2 F_0(\zeta) + zF_1(\zeta) + F_2(\zeta) + \frac{1}{z}F_3(\zeta) + \dots \quad \text{when } z^{-n} \ll \zeta$$

Thanks, Maple!

This leads to the eq. for $F_0(\zeta)$:

$$-2\beta\alpha F_0^3 + \frac{3}{2}\frac{\zeta^2 F_0'^{\,2}}{F_0} - \zeta F_0' - \zeta^2 F_0'' - \frac{1}{2}F_0 = 0$$

with gen. sol.

$$F_0(\zeta) = \frac{4C_1\zeta}{-16\alpha\beta - C_1^2(\zeta - C_2)^2}$$

Condition that $F_0(\zeta) = \zeta + O(\zeta^2) \rightsquigarrow C_2^2 = -16\,\beta\,\alpha C_1{}^{-2} - 4\,C_1{}^{-1},\ \ C_1 = \forall$
It is determined so that F_1 is a rational function (has no log) $\rightsquigarrow$

$$F_0(\zeta) = \frac{\zeta}{1 - R\zeta - N\zeta^2}$$

where $R = (A - 1 - Q)A,\ \ N = -\alpha\,\beta - 1/4\,R^2, C_1 = 4N$

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$$F_0(\zeta) = \frac{\zeta}{1 - R\zeta - N\zeta^2}$$

where $R = (A - 1 - Q)A,\ \ N = -\alpha\,\beta - 1/4\,R^2,\ C_1 = 4N$

Induction: F_n with $n \geqslant 1$ satisfy the linear non-homogeneous equations

$$\xi^2 F_n'' + \xi \left(1 - 3\xi \frac{F'}{F}\right) F_n' + \left(\frac{3}{2} \frac{\xi^2 (F')^2}{F^2} + \frac{1}{2} - 3 A^2 a F^2\right) F_n = R_n (\xi, F(\xi), F_{-1}(\xi), \dots, F_{n-1}(\xi)) \quad (1)$$

where R_n are rational functions.

Proposition

There exist constants C_1, C_2 in F_{n-1} so that $F(\xi) = \xi + O(\xi^2)$ $(\xi \to 0)$ and so that all solutions $F_n(\xi)$ are analytic at $\xi = 0$.

As a consequence, the only singularities of F_n are poles, when $N\xi^2 + R\xi - 1 = 0$, for all $n \geqslant 0$.

For example

$$F_1(\zeta) = \frac{\zeta\ (N\zeta^2 + 1)}{(N\zeta^2 + R\zeta - 1)^2} \left[-\frac{(2\,N\zeta + R)}{2\,(N\zeta^2 + 1)\,N} C_3 + C_4\right]$$

Next : a contractive argument is used to show that there is a solution asymptotic to this expansion.

Position of the poles for P_v

Solve for z:
$N\zeta^2 + R\zeta - 1 = 0, \ \zeta = Ce^{-z}z^{-Q-2}, \ \ |z|$ large , $\arg z \approx \pi/2$.

Further questions:

- Numerical verification would be great.
- For $C_+ = 0$ there exists tritonquee solutions, as explained for P_I. Where are their first array of poles located?
- How do the other two families of tronquee solutions behave?
- Does P_v admit entire solutions (for special parameters)?

Thank You!