# Exact solution of the classical dimer model on a triangular lattice

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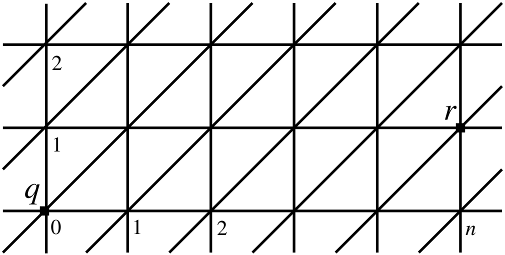
## Dimer Model

We consider the classical dimer model on a triangular lattice. It is convenient to view the triangular lattice as a square lattice with diagonals:



# Main Goal

with the weights

$$w_h = w_v = 1, \qquad w_d = t > 0.$$

Our main goal is to calculate an asymptotic behavior as $n \to \infty$ of the *monomer-monomer correlation function* $K_2(n)$ between two vertices $q$ and $r$ that are $n$ spaces apart in adjacent rows, in the thermodynamic limit (infinite volume).

When $t = 1$, the dimer model is symmetric, and when $t = 0$, it reduces to the dimer model on the square lattice, hence changing $t$ from 0 to 1 gives a deformation of the dimer model on the square lattice to the symmetric dimer model on the triangular lattice.

## Monomer-monomer correlation function as a block Toeplitz determinant

Our starting point is a determinantal formula for $K_2(n)$:

$$K_2(n) = \frac{1}{2}\sqrt{\det T_n(\phi)},$$

where $T_n(\phi)$ is the finite block Toeplitz matrix,

$$T_n(\phi) = (\phi_{j-k}), \quad 0 \le j, k \le n-1,$$

where

$$\phi_k = \frac{1}{2\pi}\int_0^{2\pi} \phi(e^{ix})e^{-ikx}dx.$$

# Block symbol $\phi(e^{ix})$

The $2 \times 2$ matrix symbol $\phi(e^{ix})$ is

$$\phi(e^{ix}) = \sigma(e^{ix}) \begin{pmatrix} p(e^{ix}) & q(e^{ix}) \\ q(e^{-ix}) & p(e^{-ix}) \end{pmatrix},$$

with

$$\sigma(e^{ix}) = \frac{1}{(1 - 2t\cos x + t^2)\sqrt{t^2 + \sin^2 x + \sin^4 x}}$$

and

$$\begin{aligned} p(e^{ix}) &= (t\cos x + \sin^2 x)(t - e^{ix}), \\ q(e^{ix}) &= \sin x(1 - 2t\cos x + t^2). \end{aligned}$$

## The exact solution of the dimer model by Kasteleyn

The exact solution of the dimer model begins with the works of *Kasteleyn* in the earlier '60s. Kasteleyn finds an expression for the partition function $Z = Z_{MN}$ of the dimer model on the *square lattice* on the rectangle $M \times N$ with free boundary conditions as a *Pfaffian* of the *Kasteleyn matrix* $A^{\mathrm{K}}$ of the size $MN \times MN$,

$$Z = \mathrm{Pf}\, A^{\mathrm{K}}\,.$$

## Diagonalization of the Kasteleyn matrix

On the square lattice the Kasteleyn matrix $A^{\mathrm{K}}$ can be explicitly block-diagonalized with $2 \times 2$ blocks along diagonal, and this gives a formula for the free energy, as a double integral of the logarithm of the spectral function.
The spectral function is an analytic periodic function which vanishes at some points, and this is a manifestation of the fact that the dimer model on a square lattice is *critical*.

# Periodic boundary conditions

**The exact solution of the dimer model with periodic boundary conditions**

Kasteleyn shows that a Pfaffian formula for the partition function is valid for the dimer model on *any planar graph*, and also he shows that the partition function of the dimer model with *periodic boundary conditions* is equal to the algebraic sum of four Pfaffians:

$$Z = \frac{1}{2}\left(-\mathrm{Pf}\, A_1^{\mathrm{K}} + \mathrm{Pf}\, A_2^{\mathrm{K}} + \mathrm{Pf}\, A_3^{\mathrm{K}} + \mathrm{Pf}\, A_4^{\mathrm{K}}\right).$$

**The work of Fisher and Stephenson**

*Fisher and Stephenson* in 1963 derive a brilliant formula for the monomer-monomer correlation function of the dimer model on the square lattice $K_2(n)$ along a coordinate axis or a diagonal, in terms of a Toeplitz determinant with the symbol

$$a(\theta) = \operatorname{sgn}\{\cos\theta\} \exp\left[-i\cot^{-1}(\tau\cos\theta)\right],$$

with jumps at $\pm\frac{\pi}{2}$ and $\beta = \frac{1}{2}$.

**The work of Fisher and Stephenson**

Fisher and Stephenson apply then a heuristic argument to show that

$$K_2(n) = \frac{B\big(1 + o(1)\big)}{n^{\frac{1}{2}}}, \quad n \to \infty.$$

A rigorous proof of this asymptotics, with an explicit constant $B > 0$, follows from a general theorem of *Deift, Its, and Krasovsky* on the Toeplitz determinants of the Fisher–Hartwig type (see also the earlier paper of *Ehrhardt*).

The polynomial decay of the correlation function indicates that the dimer model on the square lattice exhibits a *critical behavior*.

### The work of Fendley, Moessner, and Sondhi

In 2002 *Fendley, Moessner, and Sondhi* use the method of Fisher and Stephenson to derive a determinantal formula for the monomer-monomer correlation function on the triangular lattice, but are unable to analyze its asymptotics. So they ask Basor how to find the asymptotics of their determinant.

## The work of Basor and Ehrhardt

*Basor*, together with *Ehrhardt*, first rewrite the determinantal formula of Fendley, Moessner, and Sondhi as a block Toeplitz determinant, and then they find a nice explicit formula for the *order parameter*, by using Widom's extension of the Szegő theorem to block Toeplitz determinants. Let us describe the result of Basor and Ehrhardt in terms of the block Toeplitz generalization of the Borodin–Okounkov–Case–Geronimo formula.

# BOCG formula

To evaluate the asymptotics of $\det T_n(\phi)$ as $n \to \infty$ we use a *Borodin–Okounkov–Case–Geronimo* (BOCG) type formula for block Toeplitz determinants. For any matrix-valued $2\pi$-periodic matrix-valued function $\varphi(e^{ix})$ consider the corresponding semi-infinite matrices, Toeplitz and Hankel,

$$T(\varphi) = (\varphi_{j-k})_{j,k=0}^{\infty}\,; \qquad H(\varphi) = (\varphi_{j+k+1})_{j,k=0}^{\infty}\,,$$

where

$$\varphi_k = \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{ix})e^{-ikx}dx$$

# BOCG formula

Let $\psi(e^{ix}) = \phi^{-1}(e^{ix})$, where the matrix symbol $\phi(e^{ix})$ was introduced before, and the inverse is the matrix inverse. Then the following BOCG type formula holds:

$$\det T_n(\phi) = \frac{E(\psi)}{G(\psi)^n} \det(I - \Phi),$$

where $\det(I - \Phi)$ is the Fredholm determinant with

$$\Phi = H\big(e^{-inx}\psi(e^{ix})\big)\, T^{-1}\big(\psi(e^{-ix})\big)\, H\big(e^{-inx}\psi(e^{-ix})\big)\, T^{-1}\big(\psi(e^{ix})\big).$$

In our case $G(\psi) = 1$ and

$$E(\psi) = \frac{t}{2t(2+t^2) + (1+2t^2)\sqrt{2+t^2}}$$

(the *Basor–Ehrhardt* formula).

# Order parameter

The Basor–Ehrhardt formula implies that the order parameter is equal to

$$K_2(\infty) := \lim_{n\to\infty} K_2(n) = \frac{1}{2}\sqrt{E(\psi)}$$
$$= \frac{1}{2}\sqrt{\frac{t}{2t(2+t^2)+(1+2t^2)\sqrt{2+t^2}}}.$$

Our goal is to evaluate an asymptotic behavior of $K_2(n)$ as $n \to \infty$. The problem reduces to evaluating an asymptotic behavior of the Fredholm determinant $\det(I-\Phi)$, because

$$K_2(n) = K_2(\infty)\sqrt{\det(I-\Phi)}.$$

# The Wiener–Hopf factorization of $\phi(z)$

To evaluate $\det(I-\Phi)$ we need to invert the semi-infinite Toeplitz matrices $T^{-1}(\psi(e^{ix}))$ and to do so we use the *Wiener–Hopf factorization* of the symbol $\phi$. Let $z=e^{ix}$. Denote

$$\pi(z)=\begin{pmatrix} p(z) & q(z) \\ q(z^{-1}) & p(z^{-1}) \end{pmatrix},$$

so that

$$\phi(z)=\sigma(z)\pi(z),$$

where

$$\sigma(z)=\frac{1}{(1-2t\cos x+t^2)\sqrt{t^2+\sin^2 x+\sin^4 x}}$$

is a scalar function.

## The Wiener–Hopf factorization

Our goal is to factor the matrix-valued symbol $\phi(z)$ as $\phi(z) = \phi_+(z)\phi_-(z)$, where $\phi_+(z)$ and $\phi_-(z^{-1})$ are *analytic invertible* matrix valued functions on the disk $D = \{z \mid |z| \leq 1\}$. Denote

$$\tau = \frac{1}{t}\,.$$

We start with an explicit factorization of the function $t^2 + \sin^2 x + \sin^4 x$.

# Factorization of $t^2 + \sin^2 x + \sin^4 x$ and numbers $\eta_{1,2}$

We have that

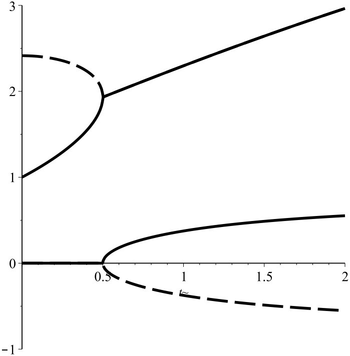
$$t^2+\sin^2 x+\sin^4 x = \frac{1}{16\eta_1^2\eta_2^2}\left(z^{-2}-\eta_1^2\right)\left(z^{-2}-\eta_2^2\right)\left(z^2-\eta_1^2\right)\left(z^2-\eta_2^2\right),$$

where

$$\eta_{1,2} = \frac{1}{\sqrt{2\pm\mu-2\sqrt{1-t^2\pm\mu}}}, \quad \mu = \sqrt{1-4t^2}.$$

The numbers $\eta_{1,2}$ are positive for $0 \le t \le \frac{1}{2}$ and complex conjugate for $t > \frac{1}{2}$.

# Graphs of $\eta_1$, $\eta_2$




The graphs of $|\eta_1(t)|$ (dashed line), $|\eta_2(t)|$ (solid line), the upper graphs, and $\arg \eta_1(t)$ (dashed line), $\arg \eta_2(t)$ (solid line), the lower graphs

**Theorem 1.** We have the Wiener–Hopf factorization:

$$\phi(z) = \phi_+(z)\phi_-(z),$$

where

$$\phi_+(z) = A(z)\Psi(z), \quad \phi_-(z) = \Psi^{-1}(z^{-1}),$$

with

$$A(z) = \frac{\tau}{z - \tau},$$

and

$$\Psi(z) = \frac{1}{\sqrt{f(z)}} D_0(z)P_1D_1(z)P_2D_2(z)P_3D_3(z)P_4D_4(z)P_5,$$

with

# Wiener–Hopf factorization

$$f(z) = \frac{(z^2 - \eta_1^2)(z^2 - \eta_2^2)}{4\eta_1\eta_2}$$

and

$$D_0(z) = \begin{pmatrix} 1 & 0 \\ 0 & z - \tau \end{pmatrix},$$

$$D_1(z) = \begin{pmatrix} z - \eta_1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_2(z) = \begin{pmatrix} z + \eta_1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$D_3(z) = \begin{pmatrix} z - \eta_2 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_4(z) = \begin{pmatrix} 1 & 0 \\ 0 & z + \eta_2 \end{pmatrix},$$

and

$$P_j = \begin{pmatrix} 1 & p_j \\ 0 & 1 \end{pmatrix}, \quad j = 1, 2, 3, 5; \quad P_4 = \begin{pmatrix} 1 & 0 \\ p_4 & 1 \end{pmatrix}.$$

# Wiener–Hopf factorization

Here

$$p_1 = \frac{i\left[\tau(\eta_1^2-1)^2 - 2\eta_1(\eta_1^2+1)\right]}{2(\eta_1^2-1)}\,, \quad p_2 = -\frac{i(\eta_1^2+1)}{\eta_1^2-1}\,,$$

$$p_3 = \frac{i\tau(\eta_1+1)}{2\eta_1}\,, \quad p_4 = -\frac{2i\eta_1\eta_2}{\tau}\,, \quad p_5 = -\frac{i\tau}{2\eta_1}\,.$$

## Idea of the proof

The idea of the proof goes back to the works of *McCoy and Wu* on the Ising model, and even before to the works of *Hopf* and *Grothendieck*.
Let us recall that $\phi(z) = \sigma(z)\pi(z)$, where $\sigma(z)$ is a scalar function. The difficult part is to factor $\pi(z)$. To factor $\pi(z)$ we use a *decreasing power algorithm*. In this algorithm at every step we make a substitution decreasing the power in $z$ of the matrix entries under consideration.

## First step

As the first step, we write $\pi(z)$ as

$$\pi(z) = \tau^{-2} \begin{pmatrix} 1 & 0 \\ 0 & z-\tau \end{pmatrix} \rho(z) \begin{pmatrix} z^{-1}-\tau & 0 \\ 0 & 1 \end{pmatrix}.$$

where

$$\rho(z) = \begin{pmatrix} \rho_{11}(z) & \rho_{12}(z) \\ \rho_{21}(z) & \rho_{22}(z) \end{pmatrix}$$

with

# First step

$$\rho_{11}(z) = z(\cos x + \tau \sin^2 x) = -\frac{\tau z^3}{4} + \frac{z^2}{2} + \frac{\tau z}{2} + \frac{1}{2} - \frac{\tau}{4z},$$

$$\rho_{12}(z) = \sin x\,(z-\tau)(z^{-1}-\tau)$$

$$= \frac{i\tau z^2}{2} - \frac{i(\tau^2+1)z}{2} + \frac{i(\tau^2+1)}{2z} - \frac{i\tau}{2z^2},$$

$$\rho_{21}(z) = -\sin x = \frac{i(z^2-1)}{2z} = \frac{iz}{2} - \frac{i}{2z},$$

$$\rho_{22}(z) = z^{-1}(\cos x + \tau \sin^2 x) = -\frac{\tau z}{4} + \frac{1}{2} + \frac{\tau}{2z} + \frac{1}{2z^2} - \frac{\tau}{4z^3}.$$

Let us factor $\rho(z)$. Observe that

$$\det \rho(z) = \frac{\tau^2}{16\eta_1^2\eta_2^2}\left(z^{-2}-\eta_1^2\right)\left(z^{-2}-\eta_2^2\right)\left(z^2-\eta_1^2\right)\left(z^2-\eta_2^2\right).$$

# Second step

Let $p_1$ be a constant. We have that

$$\begin{pmatrix} 1 & p_1 \\ 0 & 1 \end{pmatrix}^{-1} \rho(z) = \begin{pmatrix} \rho_{11}(z) - p_1\rho_{21}(z) & \rho_{12}(z) - p_1\rho_{22}(z) \\ \rho_{21}(z) & \rho_{22}(z) \end{pmatrix}.$$

Let us take

$$p_1 = \frac{\rho_{11}(\eta_1)}{\rho_{21}(\eta_1)},$$

so that

$$\rho_{11}(\eta_1) - p_1\rho_{21}(\eta_1) = 0.$$

Then, since $\det \rho(\eta_1) = 0$ and $\rho_{21}(\eta_1) \neq 0$, automatically

$$\rho_{12}(\eta_1) - p_1\rho_{22}(\eta_1) = 0,$$

and $(z - \eta_1)$ can be factored out in the first row.

# Second step

As a result we obtain that

$$\rho(z) = \begin{pmatrix} 1 & p_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z - \eta_1 & 0 \\ 0 & 1 \end{pmatrix} \rho^{(1)}(z).$$

where $\rho^{(1)}(z)$ is a rational matrix valued function with leading terms at infinity

$$\rho^{(1)}_{11}(z) = -\frac{\tau z^2}{4} + \mathcal{O}(z), \quad \rho^{(1)}_{12}(z) \ = \frac{i\tau z}{2} + \mathcal{O}(1)\,.$$

Observe that the degrees of the functions $\rho^{(1)}_{11}(z)$, $\rho^{(1)}_{12}(z)$ in $z$ decrease by one comparing to $\rho_{11}(z)$, $\rho_{12}(z)$.

# Proof of Theorem 1

We repeat this factorization procedure several times, to different rows, and, as a result, we obtain the desired explicit Wiener–Hopf factorization of the symbol. This finishes the proof of Theorem 1.

# Minus-plus factorization of $\phi(z)$

Applying the symmetry relation,

$$\phi(z) = \sigma_3 \phi^{\mathrm{T}}(z) \sigma_3,$$

to the plus-minus factorization of $\phi(z)$,

$$\phi(z) = \phi_+(z)\phi_-(z),$$

we obtain a minus-plus factorization of $\phi(z)$:

$$\phi(z) = \theta_-(z)\theta_+(z),$$

where

$$\theta_-(z) = \sigma_3 \phi_-^{\mathrm{T}}(z), \quad \theta_+(z) = \phi_+^{\mathrm{T}}(z)\sigma_3.$$

# A useful formula for the Fredholm determinant $\det(I - \Phi)$

Our goal is to evaluate the Fredholm determinant $\det(I - \Phi)$, with

$$\Phi = H\big(e^{-inx}\psi(e^{ix})\big)\,T^{-1}\big(\psi(e^{-ix})\big)\,H\big(e^{-inx}\psi(e^{-ix})\big)\,T^{-1}\big(\psi(e^{ix})\big).$$

This $\Phi$ is not very handy for an asymptotic analysis. We have another useful representation of $\det(I - \Phi)$:

$$\det(I - \Phi) = \det(I - \Lambda),$$

where

$$\Lambda = H(z^{-n}\alpha)H(z^{-n}\beta)$$

with

$$\alpha(z) = \phi_-(z)\theta_+^{-1}(z), \quad \beta(z) = \theta_-^{-1}(z^{-1})\phi_+(z^{-1}).$$

# The matrix elements of the matrix $\Lambda$

The matrix elements of the matrix $\Lambda$ are

$$\Lambda_{jk} = \sum_{a=0}^{\infty} \alpha_{j+n+a+1}\beta_{k+n+a+1},$$

where

$$\alpha_k = \frac{1}{2\pi}\int_0^{2\pi} \alpha(e^{ix})e^{-ikx}dx, \quad \beta_k = \frac{1}{2\pi}\int_0^{2\pi} \beta(e^{ix})e^{-ikx}dx.$$

We point out that this representation allows for a more direct computation of the determinant of interest without the more complicated formula involving the operator inverses.

The following theorem gives the asymptotics of the coefficients $\alpha_k$, $\beta_k$:

**Theorem 2.** Assume that $0 < t < \frac{1}{2}$. Then as $k \to \infty$, $\alpha_k$, $\beta_k$ admit the asymptotic expansions

$$\alpha_k \sim \frac{e^{-k \ln \eta_2}}{\sqrt{k}} \sum_{j=0}^{\infty} \frac{a_j^0 + (-1)^k a_j^1}{k^j},$$

$$\beta_k \sim \frac{e^{-k \ln \eta_2}}{\sqrt{k}} \sum_{j=0}^{\infty} \frac{b_j^0 + (-1)^k b_j^1}{k^j}.$$

**Asymptotics of the monomer-monomer correlation function for $0 < t < \frac{1}{2}$ .**

**Theorem 3.** Let $0 < t < \frac{1}{2}$ . Then as $n \to \infty$,

$$K_2(n) = K_2(\infty)\left[1 - \frac{e^{-2n \ln \eta_2}}{2n}\left(C_1 + (-1)^{n+1} C_2 + \mathcal{O}(n^{-1})\right)\right],$$

with some explicit $C_1, C_2 > 0$.

**Corollary.** This gives that the *correlation length* is equal to

$$\xi = \frac{1}{2 \ln \eta_2}\,.$$

As $t \to 0$,

$$\xi = \frac{1}{2t} + \mathcal{O}(1)\,.$$

# Asymptotics of the monomer-monomer correlation function for $\frac{1}{2} < t < 1$.

**Asymptotics of the monomer-monomer correlation function for $\frac{1}{2} < t < 1$.**

If $t > \frac{1}{2}$, then $\eta_1$, $\eta_2$ are complex conjugate numbers,

$$\eta_1 = e^{s-i\theta}, \quad \eta_2 = e^{s+i\theta};$$
$$s = \ln|\eta_1| = \ln|\eta_2| > 0; \quad 0 < \theta < \frac{\pi}{4}.$$

# Asymptotics of the monomer-monomer correlation function for $\frac{1}{2} < t < 1$.

The following theorem gives the asymptotics of the coefficients $\alpha_k$, $\beta_k$ in the supercritical case, $\frac{1}{2} < t < 1$:

**Theorem 4.** Assume that $\frac{1}{2} < t < 1$. Then

$$\alpha_k = \sum_{p=1,2;\,\sigma=\pm 1} \alpha_k(p,\sigma), \quad \beta_k = \sum_{p=1,2;\,\sigma=\pm 1} \beta_k(p,\sigma),$$

where as $k \to \infty$, $\alpha_k(p,\sigma)$, $\beta_k(p,\sigma)$ admit the asymptotic expansions

$$\alpha_k(p,\sigma) \sim \frac{\sigma^k e^{-k \ln \eta_p}}{\sqrt{k}} \sum_{j=0}^{\infty} \frac{a_j(p,\sigma)}{k^j},$$

$$\beta_k(p,\sigma) \sim \frac{\sigma^k e^{-k \ln \eta_p}}{\sqrt{k}} \sum_{j=0}^{\infty} \frac{b_j(p,\sigma)}{k^j}.$$

# Asymptotics of the monomer-monomer correlation function for $\frac{1}{2} < t < 1$.

**Theorem 4.** Assume that $\frac{1}{2} < t < 1$. Then as $n \to \infty$,

$$K_2(n) = K_2(\infty)\left[1 - \frac{e^{-2ns}}{2n}\Big(C_1\cos(2\theta n + \varphi_1) + C_2(-1)^n\cos(2\theta n + \varphi_2) + C_3 + C_4(-1)^n\Big) + \mathcal{O}(n^{-1})\right],$$

with $s = \ln|\eta_1| = \ln|\eta_2|$, $\theta = |\arg\eta_1| = |\arg\eta_2|$, and explicit $C_1$, $C_2$, $C_3$ , $C_4$, $\varphi_1$, $\varphi_2$.

**Conclusion**

In this work we obtain the asymptotics of the monomer-monomer correlation function for subcritical, $0 < t < \frac{1}{2}$, and supercritical, $\frac{1}{2} < t < 1$, cases:

$$K_2(n) = K_2(\infty)\left[1 - \frac{e^{-2n\ln\eta_2}}{2n}\left(C_1 + (-1)^{n+1}C_2 + \mathcal{O}(n^{-1})\right)\right],$$

and

$$K_2(n) = K_2(\infty)\left[1 - \frac{e^{-2ns}}{2n}\left(C_1\cos(2\theta n + \varphi_1)\right.\right.$$
$$\left.\left. + C_2(-1)^n\cos(2\theta n + \varphi_2) + C_3 + C_4(-1)^n\right) + \mathcal{O}(n^{-1})\right],$$

**Open problems**

1. *Double scaling limit as $t \to 0$ and $n \to \infty$.* We expect that the double scaling asymtpotics of the monomer-monomer correlation function as $t \to 0$ and $n \to \infty$ is expressed in terms of a solution to $P_{\mathrm{III}}$.
2. *Asymtpotics of the monomer-monomer correlation function at $t = \frac{1}{2}$.*
3. *Asymtpotics of the monomer-monomer correlation function for $t > 1$.* The Wiener–Hopf factorization exhibit indices.
4. *Critical asymptotics at $t = 1$.*

# Reference

**Reference**

E. Basor and P. Bleher, *Exact solution of the classical dimer model on a triangular lattice: monomer-monomer correlations.* ArXiv: 1610.08021 (to appear in Commun. Math. Phys.).

# Thank you!

**The End**

**Thank you!**