1. (a) State the definition of a CW complex, and its topology (the weak topology).

(b) Let $X$ be a CW complex and $A \subseteq X$ a nonempty CW subcomplex. Working directly from your definition, describe a CW complex structure on the quotient space $X / A$, and verify explicitly that the quotient topology on $X / A$ agrees with the weak topology of your given CW complex structure.

2. (a) Let $X$ be a path-connected, locally path-connected, and semi-locally simply connected space. Let $p : (\tilde{X}, \tilde{v}) \to (X, v)$ be the covering space associated to a subgroup $H \subseteq \pi_1(X, v)$. For an element $[\gamma] \in \pi_1(X, v)$, let $\tilde{\gamma}$ denote the lift of $\gamma$ to $\tilde{X}$ starting at $\tilde{v}$. Show that $[\gamma] \in \pi_1(X, v)$ is in the normalizer $N(H)$ of $H$ if and only if the lift $\tilde{\gamma}$ has endpoint $\tilde{w} := \tilde{\gamma}(1)$ in the orbit of $\tilde{v}$ under the deck group of the cover $p$.

(b) Consider the wedge $S^1 \vee S^1$ of circles $a$ and $b$ with wedge point $v$. Below is a (based) cover associated to a certain subgroup $H$ of $\pi_1(S^1 \vee S^1, v)$. The covering map is specified by the edge labels and orientations, and a basepoint $\tilde{v}$ is marked with a gray dot. Find a (not necessarily free) finite generating set for the normalizer $N(H)$ of $H$, with very brief justification.

3. Fix $g \geq 0$. The closed orientable genus-$g$ surface $\Sigma_g$ is the boundary of a compact 3-dimensional manifold $H_g$ called a genus-$g$ handlebody, as pictured for $g = 3$. [Image by Oleg Alexandrov]

The doubled handlebody $D_g$ is obtained by gluing two copies of $H_g$ along their boundary via the identity map. Concretely, for $H = H' = H_g$ and $I : H \to H'$ the the identity map, the space $D_g$ is the quotient of the disjoint union $H' \cup H$ by the equivalence relation $I(x) \sim x$ for all $x \in \partial H = \Sigma_g$.

(a) Compute $\pi_1(D_g)$.

(b) Compute $\tilde{H}_2(D_g)$.

For this question, you can assert descriptions of the fundamental groups and homology groups of $\Sigma_g$ and $H_g$ without proof. Please justify the other steps in your computation.
4. The following proposition is a step in the proof of the Five Lemma. Perform a diagram chase to prove this proposition.

**Proposition.** Suppose that in the following commutative diagram of abelian groups,

- Both rows are exact.
- The maps $\beta$ and $\delta$ are injective.
- The map $\alpha$ is surjective.

\[
\begin{array}{cccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\
\downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \downarrow{\delta} \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D'
\end{array}
\]

Then the map $\gamma$ is injective.

5. Let $f : X \to Y$ be a continuous map of nonempty topological spaces. Let $[0, 1]$ denote the closed interval.

The **mapping cylinder** $M_f$ of $f$ is obtained by gluing $X \times [0, 1]$ to $Y$ via $f$ in the following sense: it is the quotient of the disjoint union of $X \times [0, 1]$ and $Y$ by the equivalence relation generated by $(x, 1) \sim f(x)$.

Let $X_0$ denote the image of $X \times \{0\}$ in $M_f$. The **mapping cone** $C_f$ of $f$ is the quotient of $M_f$ that collapses $X_0$ to a point.

The spaces $M_f$ and $C_f$, respectively, are illustrated below. [Images by Fernando Muro]

Fix $k \geq 0$ in $\mathbb{Z}$. Prove that the induced map $f_* : H_i(X) \to H_i(Y)$ is an isomorphism for $0 \leq i \leq k$ if $H_i(C_f) = 0$ for $0 \leq i \leq k + 1$.

**Hint:** First verify that $(M_f, X_0)$ is a good pair.