Differential Topology QR Exam – Tuesday, January 4, 2022

All manifolds are assumed to be smooth. $\Omega^k(M)$ denotes the space of smooth k-forms and $\mathfrak{X}(M)$ the space of smooth vector fields on the manifold M.

All items will be graded independently of each other.

Problem 1. Define $F : S^2 \to \mathbb{R}^4$ by $F(x, y, z) = (x^2 - y^2, xy, xz, yz)$. Show that F induces a smooth embedding $G : \mathbb{RP}^2 \to \mathbb{R}^4$. **Note:** After you explain how a map G is obtained, to save time, you do not have to prove in detail that it is injective.

Problem 2. Let $\pi : M \to B$ be a surjective submersion.

1. Let us call a vector field $V \in \mathfrak{X}(M)$ *vertical* if and only if $d\pi_p(V_p) = 0$ for all $p \in M$. Show that if a given $X \in \mathfrak{X}(M)$ is π-related to some field $Y \in \mathfrak{X}(B)$, then for all vertical fields V the commutator $[X, V]$ is vertical.
2. Show that if $X \in \mathfrak{X}(M)$ has the following property:

$$\forall b \in B,\ \forall p, q \in \pi^{-1}(b) \qquad d\pi_p(X_p) = d\pi_q(X_q) \tag{$\heartsuit$}$$

then X is π-related to a unique *smooth* field $Y \in \mathfrak{X}(B)$.

Problem 3. Let $P = \left\{ p = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a > 0,\ b \in \mathbb{R} \right\}$.

1. Show that P is a Lie subgroup of $\mathrm{GL}(2, \mathbb{R})$, and identify its Lie algebra $T_I P$ (where I is the identity matrix).
2. Let $F : \mathrm{SO}(2) \times P \to \mathrm{SL}(2, \mathbb{R})$ be given by $F(k, p) = kp$ (matrix multiplication). Obtain a description of $dF_{(k,p)}$ that allows you to show that F is a local diffeomorphism. (F is in fact bijective and therefore a diffeomorphism, but you do not have to prove that.)

Problem 4. Let $X \in \mathfrak{X}(M)$ be a complete vector field, and $\forall t \in \mathbb{R}$ let $\theta_t : M \to M$ be the time t map of its flow. Let $\omega \in \Omega^k(M)$.

1. Recall the definition of $\mathcal{L}_X \omega$, and show that

$$\forall t \in \mathbb{R} \qquad \theta_t^* \omega = \omega \tag{$\diamondsuit$}$$

is equivalent to $\mathcal{L}_X \omega = 0$.

2. Take now $M = \mathbb{R}^n$, $\omega = dx^1 \wedge \cdots \wedge dx^n$ the standard volume form, and $X = \nabla f$ for some $f \in C^\infty(\mathbb{R}^n)$ (the usual gradient field of f). Derive a condition on f equivalent to ($\diamondsuit$).

Problem 5. Let $F : M \to N$ be a smooth map between compact, connected, oriented manifolds without boundary, of the same dimension n.

1. Let $q \in N$ be a regular value of F. Show that $\exists V \subset N$ neighborhood of q and $\forall p \in F^{-1}(q)\ \exists\, U_p \subset M$ neighborhood of p such that (i) $F^{-1}(V) = \coprod_{p \in F^{-1}(q)} U_p$ (disjoint union) and (ii) $\forall p \in F^{-1}(q)$ the restriction $F|_{U_p}$ is a diffeomorphism from U_p onto V.

2. Define $\forall p \in F^{-1}(q)$

$$(-1)^p := \begin{cases} +1 & \text{if } dF_p \text{ is orientation preserving,} \\ -1 & \text{if } dF_p \text{ is orientation reversing,} \end{cases}$$

   and let $\delta(F) = \sum_{p \in F^{-1}(q)} (-1)^p \in \mathbb{Z}$.

   Construct $\nu \in \Omega^n(N)$ supported in the neighborhood V of part (1) and such that $\int_N \nu = 1$, and prove that

$$\int_M F^* \nu = \delta(F).$$

3. Given that $H^n(M) \cong \mathbb{R} \cong H^n(N)$, deduce from (2) that the integer $\delta(F)$ is independent of the choice of q.