Algebraic Topology QR Exam – January 2022

1. (a) State the definition of a *CW complex* and its topology (the *weak topology*).

   (b) Define the 2-sphere S^2, up to homeomorphism, to be the set

   $$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$$

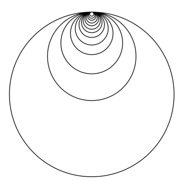
   topologized as a subspace of Euclidean 3-space. Give a rigorous proof that the 2-sphere admits a CW complex structure, verifying that its topology agrees with the weak topology on your chosen CW complex. [You may take for granted standard results from point-set topology, and standard results about continuity of maps between subspaces of Euclidean space. For the remainder of the exam you may take for granted standard results about CW complex structures on common spaces.]

2. Let G be a graph, that is, a 1-dimensional CW complex. Let S^2 denote the 2-sphere. For each of the following statements, either prove the statement, or give (with justification) a counterexample.

   (a) Every continuous map $G \to S^2$ is nullhomotopic.

   (b) Every continuous map $S^2 \to G$ is nullhomotopic.

3. For $n \in \mathbb{N}$, let C_n be the metric circle of radius $\frac{1}{n}$ in $\mathbb{R}^2$ with its north pole at the origin $(0, 0)$. Let C be the union $\bigcup_n C_n$, topologized as a subspace of Euclidean 2-space. The space C has been called the *infinite earring*, the *Hawaiian earring*, and the *shrinking wedge of circles.*

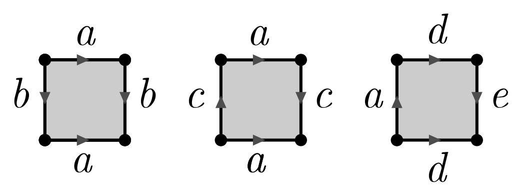
   The space C is a standard example of a space that is not *semi-locally simply connected.* Prove that C does not have a universal cover by verifying that it is not semi-locally simply connected, and proving that every space with a universal cover is semi-locally simply connected.

4. Suppose that a certain space X decomposes as the union of three open subsets, $X = U_1 \cup U_2 \cup U_3$, satisfying the following properties.

   - The open sets U_1, U_2, and U_3 are contractible.
   - The pairwise intersections $U_1 \cap U_2$, $U_1 \cap U_3$, and $U_2 \cap U_3$ are contractible.
   - The triple intersection $U_1 \cap U_2 \cap U_3$ is empty.

   Prove that X has the same homology as the circle S^1.

5. A space Y is constructed by gluing together a torus, a Klein bottle, and a cylinder along the edges labelled a below, i.e., Y is constructed from three squares using the edge identifications shown.

   (a) Calculate a presentation for the fundamental group of Y.

   (b) Calculate the homology of Y.