Differential Topology QR Exam – With Solutions
Monday, January 8, 2024

All manifolds are assumed to be smooth. $\Omega^k(M)$ denotes the space of smooth k-forms on the manifold M. All items will be graded independently of each other.

Problem 1. Let $f : X \to M$ be an injective immersion, where X and M are manifolds without boundary.

(a) Give an example, with proofs, where f is not an embedding.
(b) Show that if X is compact f must be an embedding.

Problem 2. Let M be an n-dimensional manifold. The orientation covering of M is defined as

$$\widetilde{M} = \{(p, \mathfrak{o}) \mid p \in M \text{ and } \mathfrak{o} \text{ is an orientation of } T_pM\}.$$

$\widetilde{M}$ has a C^∞ manifold structure such that the natural projection $\pi : \widetilde{M} \to M$ is a smooth covering map (you can freely use this without proof).

(a) Show that $\widetilde{M}$ has a natural orientation.
(b) Let ω be a compactly-supported n-form on M. Show that $\int_{\widetilde{M}} \pi^*\omega = 0$.

Problem 3. Let $f : X \to M$ and $g : Y \to M$ be smooth maps between manifolds, where f is a submersion. Show that

$$W := \{(x, y) \in X \times Y \mid \ f(x) = g(y)\}$$

is a submanifold of $X \times Y$. Hint: Consider $F := f \times g : X \times Y \to M \times M$.

Problem 4. Consider $\phi_t : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\phi_t(x, y, z) = (e^t x,\ \cos(t)y - \sin(t)z,\ \sin(t)y + \cos(t)z), \quad t \in \mathbb{R}.$$

(a) Show that ϕ is a flow, and find the vector field V that generates it.
(b) Use the definition of the Lie derivative of a form to compute $\mathcal{L}_V(dx \wedge dy)$.
(c) Quote Cartan's formula, and use it to verify your answer to (b).

Problem 5. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ that we identify with T_IG. Let Ω^k_G denote the space of all left-invariant forms on G of degree k.

(a) Establish a natural isomorphism $\Omega^k_G \cong \bigwedge^k \mathfrak{g}^*$.
(b) Show that the exterior differential maps Ω^k_G into Ω^{k+1}_G.
(c) Combining (a) and (b) with $k = 0, 1$, we obtain maps

$$d_0 : \wedge^0\mathfrak{g}^* \cong \mathbb{R} \to \mathfrak{g}^* \quad \text{and} \quad d_1 : \mathfrak{g}^* \to \wedge^2\mathfrak{g}^*.$$

Show that $d_0 = 0$ and compute d_1. Hint: For d_1, use a formula for $d\alpha(V, W)$ where α is any one-form and V, W are vector fields.