General and Differential Topology QR Exam – May 4, 2023

All manifolds, vector fields, and differential forms are assumed to be smooth (C^∞).

SOLUTIONS

Problem 1. Let $M = \{(w, x, y, z) \in \mathbb{R}^4 \mid w^2 + x^2 = y^2 + z^2 = 1\}$.
(a) Show that M is a submanifold of $\mathbb{R}^4$.
(b) Define a diffeomorphism $\pi : M \to M$ by $\pi(w, x, y, z) = (-y, -z, w, x)$. Let G be the group generated by this diffeomorphism. Show that the orbit space M/G is a manifold.
(c) Is M/G orientable?

Solution.
(a) Let $f : \mathbb{R}^4 \to \mathbb{R}^2$ be given by $f(w, x, y, z) = (w^2 + x^2, y^2 + z^2)$. We can easily see that $(1, 1)$ is a regular value of f since we cannot simultaneously have $2w = 2x = 0, w^2 + x^2 = 1$ or $2y = 2z = 0, y^2 + z^2 = 1$, so $f^{-1}(1, 1)$ is a submanifold.
(b) We observe that $\pi^2(w, x, y, z) = (-w, -x, -y, -z)$, so π has order 4 and π^2 has no fixed points, which implies that G acts freely on M. Since G is a finite group acting via smooth maps, this implies that M/G is a manifold.
(c) Let $\alpha = (-xdw + wdx) \wedge (-zdy + ydz)$, a volume form for M. We compute that $\pi^*\alpha = -\alpha$. Since M is connected, this implies that M/G is not orientable.

□

Problem 2. Let $n \geq 2$. Let X be the set of real $n \times n$ matrices A satisfying $A + A^t = 0$, where A^t is the transpose of A.
(a) Is X a Lie algebra?
(b) Let $\mathrm{GL}(n)$ be the group of invertible $n \times n$ matrices. Is $X \cap \mathrm{GL}(n)$ a Lie group?
(c) Let $M(n)$ be the set of all real $n \times n$ matrices. Define a function $f : X \to M(n)$ by $f(A) = e^A - e^{-A}$. Describe the image under f of a small open neighborhood of the zero matrix.

Solution.
(a) Yes - we can either compute that it is the Lie algebra of $O(n)$ or check that if $A, B \in X$ then

$$[A, B] + [A, B]^t = AB - BA + B^tA^t - A^tB^t = AB - BA + BA - AB = 0.$$

(b) No - for instance it does not contain the identity matrix, so it is not even a group.
(c) We first observe that

$$f(A)^t = e^{A^t} - e^{-A^t} = e^{-A} - e^A = -f(A),$$

so f maps X into itself. We next compute the map induced by f on tangent spaces at zero: $df(B)$ is the linear part in t of $f(tB) = 2tB + O(t^3)$, so $df(B) = 2B$. Thus $f : X \to X$ is regular at zero and hence maps a small open neighborhood of the zero matrix in X to a small open neighborhood of the zero matrix in X.

□

Problem 3. Let α be a nonvanishing 1-form on a manifold M, so for any point $q \in M$, $\ker \alpha_q$ is a codimension 1 subspace of the tangent space T_qM. Assume that f is a nonvanishing smooth function on M such that

$$d(\alpha) = \frac{df}{f} \wedge \alpha.$$

Prove that for any $p \in M$, there is a regular submanifold S of M such that $p \in S$ and $T_qS = \ker \alpha_q$ for all $q \in S$.

Solution. We first observe if we had $\alpha = dh$ for some smooth function h on M without then we would be done, since we are given that α is nonvanishing and hence h has no critical points and we could then take $S = h^{-1}(h(p))$ for each $p \in S$. In fact, we only need this to be true in a neighborhood of each point p. By the Poincaré Lemma, it would thus be sufficient if we knew that $d\alpha = 0$.

But this isn't quite what we are given, so we need to be a little more general. In fact, the same argument gives that we are happy if $\alpha = gdh$ for some g, h in a neighborhood of every point, and hence by the Poincaré Lemma we just need that

$$d(\frac{1}{g}\alpha) = 0.$$

But this is exactly what we are given with $g = f$, so we are done. $\square$

Problem 4. Let X be a complete vector field on a manifold M, and let $\alpha \in \Omega^k(M)$ be a k-form.

(a) Show that the following two conditions on the pair (X, α) are equivalent:
- the Lie derivative $\mathcal{L}_X\alpha$ is identically zero;
- for all $t \in \mathbb{R}$, $\theta_t^*\alpha = \alpha$, where $\theta_t : M \to M$ is the time t map of the flow along X.

(b) Suppose that $M = \mathbb{R}^3$, $\alpha = dx \wedge dy \wedge dz$, and

$$X = ax(y-z)\frac{\partial}{\partial x} + by(z-x)\frac{\partial}{\partial y} + cz(x-y)\frac{\partial}{\partial z}$$

for some $a, b, c \in \mathbb{R}$. For which a, b, c is it the case that (X, α) satisfies the conditions of the previous part?

Solution.

(a) Recall that $\mathcal{L}_X\alpha$ is defined as the derivative with respect to t of $\theta_t^*\alpha$ at $t = 0$. This immediately gives that the second condition implies the first. In the other direction, we observe that for any $s \in \mathbb{R}$ we have

$$\left[\frac{d}{dt}\theta_t^*\alpha\right]_{t=s} = \left[\frac{d}{dt}\theta_s^*\theta_t^*\alpha\right]_{t=0} = 0.$$

(b) We know (e.g. by Cartan's magic formula) that for a volume form α, $\mathcal{L}_X\alpha = d(\iota_X\alpha)$. Computing in this case gives

$$\mathcal{L}_X\alpha = (a(y-z) + b(z-x) + c(x-y))dx \wedge dy \wedge dz,$$

which is 0 exactly when $a = b = c$.

$\square$

Problem 5. Let M be a compact manifold of positive dimension. Prove that there exists a vector field X on M such that for every nonempty open set U of M, X is not identically zero on U.

Solution. Note that on any chart for M we have a vector field that is not identically zero on any nonempty open subset (by taking dx_1). Since M is compact, we can cover it by finitely many charts $U_1, \ldots, U_n$ with corresponding vector fields $X_1, \ldots, X_n$ as above. Let $f_1, \ldots, f_n$ be a partition of unity subordinate to this cover. Then $f_1X_1, \ldots, f_nX_n$ are vector fields on M such that for every nonempty open subset, at least one of the vector fields is not identically zero.

We claim that some linear combination $X = c_1f_1X_1 + \cdots + c_nf_nX_n$ will have the desired property, for suitable constants $c_1, \ldots, c_n \in \mathbb{R}$. Indeed, we only need to check the property on a countable base for the topology of M. On each open set in this base, X will be identically zero only for $(c_1, \ldots, c_n)$ belonging to some proper linear subspace of $\mathbb{R}^n$. Since the complement of countably many proper linear subspaces in $\mathbb{R}^n$ is nonempty, we are done. $\square$