THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Algebraic Topology
Solutions

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1. The *unreduced suspension* of a topological space X is the quotient of the space $X \times [0,1]$ by the smallest equivalence relation $\sim$ which has $(x,0) \sim (y,0)$ and $(x,1) \sim (y,1)$ for all $x, y \in X$, with the quotient topology. For which $n \in \mathbb{N}$ is the unreduced suspension Z_n of the real projective space $\mathbb{R}P^n$ a topological manifold without boundary (i.e. has the property that every point $u \in Z_n$ has a neighborhood homeomorphic to $\mathbb{R}^k$ for some k)?

   Solution: The group $H_i(\mathbb{R}^k, \mathbb{R}^k \smallsetminus \{0\})$ is $\mathbb{Z}$ for $i = k$ and 0 otherwise By excision, if a space M is a topological manifold of dimension k, then for any $* \in M$ and any open set $U \ni *$, we have

   $$H_i(U, U \smallsetminus \{*\}) \cong H_i(\mathbb{R}^k, \mathbb{R}^k \smallsetminus \{0\}).$$

   In our case, we can take

   $$U = \mathbb{R}P^n \times (1/2, 1]/(x,1) \sim (y,1),$$

   which is contractible, while denoting by $*$ the point which is the image of $(x,1)$, we have

   $$U \smallsetminus \{*\} = \mathbb{R}P^n \times (1/2, 1),$$

   which is homotopy equivalent to $\mathbb{R}P^n$. So by the long exact sequence in homology,

   $$H_i(U, U \smallsetminus \{*\}) \cong \widetilde{H}_{i-1}(\mathbb{R}P^n).$$

   Since the right hand side has non-zero torsion for $n > 1$ for at least one choice of i, the answer is NO for $n > 1$. For $n = 1$, $\mathbb{R}P^1 \cong S^1$, so the answer is YES.

2. Give an example of a subgroup $H \subset F(a,b)$ of the free group on two generators a, b which has finite index but is not normal. Recalling that H is necessarily also free, give a set of free generators of H.

   Solution: We can get an example by taking an irregular cover of the graph with one vertex and two loops a, b. For example, we can use a graph with three vertices x, y, z, an a-loop on x, a b edge from x to y and back, an a-edge from y to z and back, and a b-loop on z. Taking the base point to be, say, x, we obtain generators $a, b^2, ba^2b^{-1}, baba^{-1}b^{-1}$ (but there are infinitely many other correct answers).

3. Let X be the quotient of the space $S^1 \times S^1$ obtained by identifying two different chosen points. Is the universal covering space of X contractible? Explain.

Solution: The space X is homotopy equivalent to the one-point union

$$Y = (S^1 \times S^1) \vee S^1,$$

so its fundamental group is the free product $G = (\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z}$. Denote by H the normal envelope in G of the free factor of G isomorphic to $\mathbb{Z}$. Then H is isomorphic to the free group on $\mathbb{Z} \times \mathbb{Z}$. Additionally, the based covering of Y with respect to the subgroup $H \in \pi_1(Y, *)$ (where $*$ is the point of identification of the one-point union) is $\mathbb{R}^2$ with a separate copy of S^1 attached by one point to each point of $\mathbb{Z} \times \mathbb{Z} \subset \mathbb{R}^2$. This space is homotopy-equivalent to a graph, so its universal cover is contractible. Since universal covers (say, of CW-complexes) preserve homotopy equivalence, the universal cover of X is also contractible, so the answer is YES.

4. Let X be a CW-complex with exactly four cells, of dimensions $0, n, n+1, n+2$, where $n > 0$. Assume further that the attaching map of the $(n+1)$-cell is not homotopic to a constant map. Denoting by X_n the n-skeleton of X, prove that the quotient space X/X_n is homotopy equivalent to $S^{n+1} \vee S^{n+2}$ where $Y \vee Z$ denotes the one-point union, i.e. the quotient of the disjoint union by identifying one point of Y with one point of Z. [Hint: Use the definition of cellular homology.]

Solution: Maps $f : S^k \to S^k$ for $k > 0$ are classified, up to homotopy, by what they induce on $\mathbb{Z} \cong H_k(S^k)$. This is multiplication by an integer called the *degree* $deg(f)$. Let k be the degree of the attaching map of the $(n+1)$-cell of X and let ℓ be the degree of the attaching map of the $(n+2)$-cell in X/X_n. The reduced cellular homology complex $\widetilde{C}^{cell}(X)$ then is

$$\mathbb{Z} \xrightarrow{\ \ell\ } \mathbb{Z} \xrightarrow{\ k\ } \mathbb{Z}$$

in dimensional degrees $n+2, n+1, n$. Since we must have $dd = 0$, one of the numbers k, ℓ must be 0. Since we assumed $k \neq 0$, we have $\ell = 0$. This means that the attaching map of the $(n+2)$-cell in X/X_n is homotopic to a constant map. Homotopic attaching maps produce homotopy equivalent mappng cones.

5. Let $X = (S^1 \times S^1)/(\{1, -1\} \times S^1)$ where $S^1 \subset \mathbb{C}$ is the unit circle. Calculate the homology groups of X.

Solution: The space is homotopy equivalent to the one-point union

$$S^2 \vee S^2 \vee S^1 \vee S^1,$$

so $H_0(X) \cong \mathbb{Z}$, $H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$, $H_2(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ and the other homology groups are 0.