This exam consists of five problems. All manifolds are assumed to be $C^\infty$. $\mathfrak{X}(M)$ denotes the space of smooth vector fields on the manifold $M$. All items will be graded independently of each other.

**Problem 1.** The subgroup $\Gamma$ of $SU(2)$ generated by the matrix

$$\gamma := \begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix} \in SU(2)$$

is isomorphic to $\mathbb{Z}_4$, and it acts on the unit three-sphere $S^3 \subset \mathbb{C}^2$ by matrix multiplication. Let $X = S^3/\Gamma$ be the orbit space, with the quotient topology. Answer the following questions, with proofs:

1. Is $X$ second countable?
2. Is $X$ Hausdorff?
3. Is the projection $\pi : S^3 \to X$ a local homeomorphism?

**Solution:**
1. Yes. Since the action is continuous the projection $\pi$ is an open map. Then, applying a standard result, $X$ is second countable because $S^3$ is.
2. Yes. Similarly, a standard result is that since $\pi$ is open $X$ is Hausdorff iff the orbit relation is closed in $S^3 \times S^3$. Since the group is finite and $S^3$ compact, the orbit relation is the union of finitely-many closed sets (the graphs of the action of each group element), so it is closed.
3. Yes, because in addition the action is free (which should be checked).

**Problem 2.** A smooth $F : M \to M$ is called a Lefschetz map iff for all $p \in M$ such that $F(p) = p$ one has:

$$1 \in \mathbb{R} \text{ is not an eigenvalue of } \quad dF_p : T_pM \to T_pM.$$ 

1. Show that $F$ is Lefschetz iff its graph and the diagonal $\Delta \subset M \times M$ intersect transversely.
2. Show that if $F$ is Lefschetz then the set $\{p \in M : F(p) = p\}$ consists of isolated points.
3. Is the converse of the previous statement true?

**Solution:** Let $\Gamma \subset M \times M$ be the graph of $F$; this is a regular submanifold because it is the image of the proper embedding $M \ni p \mapsto (p, F(p)) \in M \times M$. Let $(p, p) \in \Gamma \cap \Delta$, i.e. $F(p) = p$. Then

$$T_{(p,p)}\Gamma = \{(v, dF_p(v)) \mid v \in T_pM\} \quad \text{and} \quad T_{(p,p)}\Delta = \{(v, v) \mid v \in T_pM\}$$

Both of these spaces have dimension $n$, the dimension of $M$, so the intersection is transversal iff their intersection is zero, that is iff $\forall v \in T_pM \quad dF_p(v) = v \Rightarrow v = 0$. 

(2) By transversality, the intersection \( \Gamma \cap \Delta \) is a manifold of codimension \( 2n \), that is, it has dimension zero. Therefore it consists of isolated points.

(3) No, it suffices to consider a map \( F : (0,1) \to (0,1) \) with finitely-many fixed points whose graph is tangent to the diagonal at one point.

**Problem 3.** Let \( S^2 \subset \mathbb{R}^3 \) be the two-sphere with the standard orientation, and \( F : S^2 \to S^2 \) be given by \( F(a,b,c) = (a,-b,-c) \). Note that \( F \) is a diffeomorphism (no proof needed). Also, let \( x, y, z \) denote the restrictions of the coordinate functions to \( S^2 \), and let \( \alpha = xydy \wedge dz \).

1. Establish whether or not \( F \) is orientation preserving, and compute \( F^*(\alpha) \).

   What do your findings imply about the value of \( \int_{S^2} \alpha \)? Explain.

2. The vector field \( (-y, x, 0) \) in \( \mathbb{R}^3 \) is tangent to the sphere, and therefore it restricts to a vector field \( X \in \mathfrak{X}(S^2) \). Compute the Lie derivative \( \mathcal{L}_X \alpha \).

**Solution:** (1) Since the sphere is connected, it suffices to check whether \( dF_p \) is orientation preserving or not at a single point \( p \in S^2 \). The simplest choice is \( p = (1, 0, 0) \), so the tangent space is identified with the \( yz \) plane. It is clear that \( dF_p \) is multiplication by \((-1)\), which is orientation-preserving in dimension two. Moreover, since \( F^*x = x \), \( F^*y = -y \) and \( F^*z = -z \), one has \( F^*\alpha = x(-y)d(-y) \wedge d(-z) = -\alpha \). Since integration is invariant under pull-backs by orientation-preserving diffeomorphisms, \( \int_{S^2} \alpha = \int_{S^2} F^*\alpha = -\int_{S^2} \alpha \), and therefore \( \int_{S^2} \alpha = 0 \).

(2) We apply Cartan’s magic formula. Note that \( d\alpha = 0 \) by dimensional considerations, so we are left with \( \mathcal{L}_X \alpha = d_X\alpha = d(x^2ydz) = 2xydx \wedge dz + x^2dy \wedge dz \).

**Problem 4.** Let \( M \) be a smooth manifold, and \( X \in \mathfrak{X}(M \times \mathbb{R}) \) be a smooth vector field of the form

\[
\forall (p, s) \in M \times \mathbb{R} \quad X(p,s) = (V(p,s), \partial_s), \quad \text{where } V_{p,s} \in T_p M.
\]

(We are identifying \( T_{(p,s)}(M \times \mathbb{R}) \) with \( T_p M \times T_s \mathbb{R} \).) For each \((p, s) \in M \times \mathbb{R}\), let \( t \mapsto \Phi_t(p, s) \) be the integral curve of \( X \) starting at \((p, s)\), and denote

\[
\phi_{t,s}(p) := \pi(\Phi_{t-s}(p, s)), \quad \text{where } \pi : M \times \mathbb{R} \to M \text{ is the projection}.
\]

1. Show that \( \forall t_0 \in \mathbb{R}, \ p \in M \) the curve on \( M \) \( t \mapsto \gamma(t) = \phi_{t,t_0}(p) \) is defined in a neighborhood of \( t_0 \) and satisfies \( \dot{\gamma}(t) = V_{\gamma(t),t}, \gamma(t_0) = p \).

2. Assuming that \( X \) is complete, show that \( \forall r, s, t \in \mathbb{R} \)

\[
\phi_{t,s} \circ \phi_{s,r} = \phi_{t,r},
\]

where \( \phi_{t,s} : M \to M \text{ is the map } p \to \phi_{t,s}(p) \), etc.

**Solution:** (1) The integral curve of \( X, \ u \mapsto \Phi_u(p, s) \), is defined for \( u \) in a neighborhood of zero. (2) By definition, \( \Phi_{t-s}(p, s) = (\phi_{t,s}(p), t) \), or \( \Phi_u(p, s) = (\phi_{u+s,s}(p), u+s) \).
Using that $\Phi$ is a one-parameter group,

$$\Phi_t \circ \Phi_u(p, s) = \Phi_t(\phi_{u+s,s}(p), u + s) = (\phi_{t+u+s,u+s} \circ \phi_{u,s,s}(p), u + s + t) = \Phi_{t+u}(p, s),$$

or

$$\phi_{t+u+s,s}(p) = \phi_{t+u+s,u+s} \circ \phi_{u,s,s}(p).$$

A change of variables yields the desired result.

**Problem 5.** Let $G$ be a Lie group with Lie algebra $\mathfrak{g} = T_e G$ (where $e$ is the identity), and let $A, B \in \mathfrak{g}$ be linearly independent elements satisfying $[A, B] = 0$.

1. Carefully explain why $\forall s, t \in \mathbb{R}$ $\exp(sA) \exp(tB) = \exp(tB) \exp(sA)$, where $\exp : \mathfrak{g} \to G$ is the exponential map.

2. Show that the map $E : \mathbb{R}^2 \ni (s, t) \mapsto \exp(sA) \exp(tB) \in G$ is a an immersion.

3. Specialize to the case $G = U(3)$, and $A, B$ diagonal with diagonal entries the components of $i\vec{\lambda} = \langle i\lambda_1, i\lambda_2, i\lambda_3 \rangle$ and $i\vec{\mu} = \langle i\mu_1, i\mu_2, i\mu_3 \rangle$ respectively.

Under what conditions on $\vec{\lambda}, \vec{\mu} \in \mathbb{R}^3$ is the image of the map $E$ a closed (regular) submanifold of $U(3)$?

**Solution:** (1) Denote by $A^t, B^s$ the left-invariant vector fields corresponding to $A$ and $B$. By left-invariance, left multiplication by any $g \in G$ maps integral curves of $B^s$ to integral curves, and therefore

$$\exp(sA) \exp(tB) = \phi_t^{B^s}(\exp(sA)) = \phi_t^{B^s} \circ \phi_s^{A^t}(e)$$

since $\exp(sA) = \phi_s^{A^t}(e)$. A similar expression holds with $sA$ and $tB$ exchanged. On the other hand, the assumption on $A$ and $B$ is that $A^t, B^t$ commute as vector fields. Therefore, their flows $\phi^{A^t}$ and $\phi^{B^s}$ commute, and the result follows.

(2) Pick $(s, t) \in \mathbb{R}^2$ and compute

$$dE_{(s,t)}(\partial_t) = dL_{\exp(sA)}(B^s_{\exp(tB)}) = B^s_{E(s,t)} = dL_{E(s,t)}(B)$$

where $L_g : G \to G$ is left translation by $g \in G$. Similarly, $dE_{(s,t)}(\partial_s) = dL_{E(s,t)}(A)$. Since $A, B$ are linearly independent and $dL_{E(s,t)} : T_e G \to T_{E(s,t)}G$ is an isomorphism, the result follows.

(3) One computes that $E(s, t)$ is the diagonal matrix with entries $e^{i(s\lambda_j + t\mu_j)}$, $j = 1, 2, 3$. Observe that the image of $E$ is contained in the 3-torus $T \subset U(3)$ of diagonal matrices, and the question is equivalent to whether the image of $E$ is a submanifold of $T$. If $\Pi \subset \mathbb{R}^3$ is the span of $\{\vec{\lambda}, \vec{\mu}\}$, the condition is that

$$\Pi \cap 2\pi \mathbb{Z}^3$$

be a lattice in $\Pi$. 