General and Differential Topology QR Exam – Aug 17, 2022

SOLUTIONS

Problem 1. Let M be a smooth manifold.

(a) Prove that the total space of the tangent bundle TM is orientable.
(b) Now suppose M is the union of two orientable open submanifolds U_1, U_2 and the intersection $U_1 \cap U_2$ is connected. Prove that M is orientable.

Solution.

(a) Let $\{U_\alpha, \phi_\alpha : U_\alpha \to \mathbb{R}^n\}$ be any atlas for M. We then have maps $T\phi_\alpha : TU_\alpha \to T\mathbb{R}^n$. Identifying $T\mathbb{R}^n$ with $\mathbb{R}^{2n}$ (with the first n coordinates giving the point in $\mathbb{R}^n$ and the last n giving the tangent vector), we obtain an atlas $\{TU_\alpha, T\phi_\alpha : TU_\alpha \to \mathbb{R}^{2n}\}$ for TM. It remains to check that this atlas is oriented, i.e. that the determinant of the Jacobian of any transition function $T\phi_\alpha \circ T\phi_\beta^{-1}$ is positive. But if we decompose this $(2n)\times(2n)$ Jacobian matrix into $n\times n$ blocks, it is clearly block lower triangular, since the map on points doesn't depend on the chosen tangent vector. Moreover, the two blocks on the diagonal are equal since they are both the Jacobian matrix of the transition function $\phi_\alpha \circ \phi_\beta^{-1}$ for the original atlas. Therefore the determinant of this $(2n) \times (2n)$ Jacobian is the square of the determinant of the $n \times n$ Jacobian, hence is positive.
(b) Let $V = U_1 \cap U_2$. First, if V is empty then M is the disjoint union of U_1 and U_2 and we can just take any orientation on each of them to get an orientation of M.

Now assume V is nonempty. Since U_1 is orientable and V is an open submanifold of U_1, V is orientable. Since V is connected, it has exactly 2 orientations, which are opposite to each other. Thus if we choose any orientations on U_1 and U_2 and then reverse the orientation on U_2 if necessary, the induced orientations on V from U_1 and U_2 will coincide. This then gives an orientation of M.

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Problem 2. Let $M = \mathbb{R}^2/\mathbb{Z}^2$ $(= S^1 \times S^1)$. Let $\iota : M \to M$ be the involution $\iota(x, y) = (-x, y+0.5)$. Let Q be the quotient of M by ι, with a natural smooth manifold structure inherited from $\mathbb{R}^2$.

(a) Explain how to interpret dy as defining a 1-form on Q.
(b) Show that, viewed as a 1-form on Q, dy is closed but not exact (so gives a nonzero element in $H^1_{\mathrm{dR}}(Q)$).

Solution.

(a) The standard 1-form dy on $\mathbb{R}^2$ is invariant both with respect to the action of $\mathbb{Z}^2$ and the action of ι, so it descends to a 1-form on the quotient manifold Q. (If we wanted to make this more explicit, note that the quotient map $M \to Q$ has sections $U \to M$ (diffeomorphisms onto the image) for sufficiently small open sets $U \subset Q$, and picking one such section lets us define a 1-form dy on U. Invariance of dy with respect to the group action means that this doesn't depend on the choice of section U, and then it is easy to see that these 1-forms glue to a 1-form on all of Q.)
(b) To see that dy is closed as a 1-form on Q, note that $d(dy) = (d^2)y = 0$ on $\mathbb{R}^2$, and d clearly commutes with the quotient construction in the previous part since it can be computed on small open sets. To see that dy is not exact as a 1-form on Q, consider the curve $\phi : [0, 1] \to Q$ given by $\phi(t) = (0, t)$. We clearly have that the integral of dy over ϕ is $1 \neq 0$. But if dy was exact, it would be df for some function $f : Q \to \mathbb{R}$ and then the integral would be $f(\phi(1)) - f(\phi(0))$, which is 0 because $\phi(1) = \phi(0)$ in Q. Therefore dy is not exact.

□

Problem 3. Let $\mathrm{SL}(n)$ be the group of $n \times n$ real matrices with determinant 1, considered as a submanifold of the vector space $\mathcal{M}(n)$ of all $n \times n$ real matrices. For each $g \in \mathrm{SL}(n)$ identify the tangent space $T_g \mathrm{SL}(n)$ with a linear subspace of $\mathcal{M}(n)$.

(a) Explicitly compute the linear subspace $T_g \mathrm{SL}(n)$ for an arbitrary $g \in \mathrm{SL}(n)$.
(b) Let $F : \mathrm{SL}(n) \to \mathrm{SL}(n)$ be the map $F(g) = gg^T$, where g^T is the transpose of the matrix g. Explicitly compute the map on tangent spaces $dF_{\mathsf{id}} : T_{\mathsf{id}} \mathrm{SL}(n) \to T_{\mathsf{id}} \mathrm{SL}(n)$, where id is the identity matrix. What is the rank of dF_{id}?

Solution.

(a) First we compute the tangent space to $\mathrm{SL}(n)$ at the identity matrix id. This can be done by linearizing the determinant function: if $x \in \mathcal{M}(n)$, then $\det(\mathsf{id} + tx) = 1 + \mathrm{tr}(x)t + O(t^2)$. Therefore $T_{\mathsf{id}} \mathrm{SL}(n)$ is the space of traceless matrices.

For the tangent space at an arbitrary matrix $g \in \mathrm{SL}(n)$, note that $\mathrm{SL}(n)$ is a Lie group, so left multiplication by g induces an isomorphism from $T_{\mathsf{id}} \mathrm{SL}(n)$ to $T_g \mathrm{SL}(n)$. Thus $T_g \mathrm{SL}(n)$ is the space of matrices of the form $g \cdot (\text{traceless})$.

(b) To compute $dF_{\mathsf{id}}(x)$ for a tangent vector x (a traceless matrix by the previous part), we simply need to compute the coefficient of t^1 in $F(\mathsf{id} + tx) = (\mathsf{id} + tx)(\mathsf{id} + tx^T)$, which is $x + x^T$. So $dF_{\mathsf{id}}(x) = x + x^T$. It is easy to see that the image of this linear map is the space of symmetric traceless $n \times n$ matrices, which has rank $(1 + 2 + \cdots + (n-1)) + (n-1) = \frac{n^2+n-2}{2}$.

□

Problem 4. Here are two unrelated questions about submersions of smooth manifolds $\pi : M \to B$.

(a) Suppose that $\dim M = \dim B$ and M is compact. Prove that $\pi^{-1}(q)$ is a finite set for any point $q \in B$.
(b) Suppose that Y is a smooth vector field on B. Prove that there exists a smooth vector field X on M such that X is π-related to Y (i.e. $\pi_*(X(p)) = Y(\pi(p))$ for any point $p \in M$). (Hint: first show that you can find such an X locally on M, then use a partition of unity argument.)

Solution.

(a) Since $\dim M = \dim B$, the submersion π is a local diffeomorphism. In particular, the fiber $\pi^{-1}(q)$ is discrete. This fiber is also a closed subset of the compact manifold M, so it is compact. But a discrete compact topological space is finite, so we are done.

(b) Since π is a submersion, near each point in M it looks locally like the projection of an open set $U \subset \mathbb{R}^m$ under the map $p : \mathbb{R}^m \to \mathbb{R}^n$ forgetting the last $m - n$ coordinates. In such coordinates, given a vector field $Y = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ on the image $p(U)$, it is easy to find a p-related vector field X on U: we simply take X to be given by the same expression as X (with no dependence on the last $m - n$ coordinates).

Thus we can cover M by open sets U_α such that there exist vector fields X_α on U_α such that X_α is $\pi|_{U_\alpha}$-related to Y. It remains to combine these using a partition of unity argument. Let $\{f_\alpha\}$ be a partition of unity subordinate to the open cover $\{U_\alpha\}$. Then let $X = \sum_\alpha f_\alpha X_\alpha$. We then have

$$\pi_*(X(p)) = \sum_\alpha f_\alpha(p) \pi_*(X_\alpha(p)) = \sum_\alpha f_\alpha(p) Y(\pi(p)) = Y(\pi(p)),$$

where the sums run over the finitely many α with $f_\alpha(p) \neq 0$.

□

Problem 5. Let M be a compact smooth manifold. Let X be a smooth vector field on M. For each part, just give a brief explanation or brief description of a counterexample.

(a) Is X necessarily complete? (A complete vector field is one such that all integral curves extend to be defined for all $t \in \mathbb{R}$.)
(b) Now assume that X is complete, and take the integral curve through some point p, $\phi_p : \mathbb{R} \to M$. Is ϕ_p necessarily an immersion?
(c) Now assume an integral curve ϕ_p is an immersion. Is the image of ϕ_p necessarily closed in M?

Solution.

(a) Yes, X is complete. Near any point we can find an open neighborhood where the integral curve is defined on $(-\epsilon, \epsilon)$ for some $\epsilon > 0$, and then compactness of M gives that we can find such an ϵ that works uniformly on all of M, and then we can extend the domain to $\mathbb{R}$ by repeatedly extending by ϵ.
(b) No, ϕ_p is not an immersion - for a counterexample, simply take $X = 0$ so that ϕ_p is constant. (We would need to require that X be nonvanishing to get an immersion.)
(c) No, the image isn't necessarily closed in M. One way to see this is to take a translation-invariant vector field on the torus $\mathbb{R}^2/\mathbb{Z}^2$, with some irrational slope - then the image of any integral curve will be dense on the torus (but not all of the torus).

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