Differentiable Manifolds QR Exam – January 5, 2021

All manifolds are assumed to be C^∞. All items will be graded independently of each other.

Problem 1.- Let $M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \; ; \; x_1^2 + x_2^2 = 1 \text{ and } x_3^2 + x_4^2 = 1\}$, and let $\iota : M \hookrightarrow \mathbb{R}^4$ be the inclusion.

1. Show that M is a submanifold of $\mathbb{R}^4$.

2. If $\alpha = -x_2dx_1 + x_1dx_2 - x_4dx_3 + x_3dx_4$, show that $\iota^*(\alpha)$ is closed but not exact.

SOLUTION: (1) This can be proved either using the regular value theorem applied to the map $F(x_1, x_2, x_3, x_4) = (x_1^2+x_2^2, x_3^2+x_4^2)$ and the value $(1, 1)$, or by considering the immersion $G : S^1 \times S^1 \to \mathbb{R}^4$ which is injective and therefore an embedding since $S^1 \times S^1$ is compact.

(2) Parametrize M by: $x_1 = \cos(\theta_1)$, $x_2 = \sin(\theta_1)$, $x_3 = \cos(\theta_2)$, $x_4 = \sin(\theta_2)$. Then a calculation shows that in the (θ_1, θ_2) coordinates

$$\iota^*(\alpha) = d\theta_1 + d\theta_2.$$

(For example $\iota^*(-x_2dx_1 + x_1dx_2) = -\sin(\theta_1)d(\cos(\theta_1) + \cos(\theta_1)d(\sin(\theta_1)) = d\theta_1$.) Although the coordinates are defined only mod 2π the above expression for $\iota^*(\alpha)$ is global. From it it follows that $\iota^*(\alpha)$ is closed. It is not exact because e.g. if γ is the curve on M parametrized by $\theta_2 = 0, \theta_1 \in [0, 2\pi]$ then $\int_\gamma \iota^*(\alpha) = 2\pi \neq 0$, and by Stokes' theorem $\iota^*(\alpha)$ cannot be exact.

Problem 2.- Let $A \in T_I\mathrm{O}(n) \setminus \{0\}$ where I is the identity and $\mathrm{O}(n)$ the orthogonal group. Let $A^\sharp$ be the left-invariant vector field on $\mathrm{O}(n)$ whose value at the identity is A. Define $\forall t \in \mathbb{R}$

$$\exp(tA) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k.$$

1. Give a direct proof that $s \mapsto \exp(sA)$ is the integral curve of $A^\sharp$ starting at the identity.

2. Derive an expression of the time-t map $\phi_t : \mathrm{O}(n) \to \mathrm{O}(n)$ of the flow of $A^\sharp$ in terms of $\exp(tA)$.

SOLUTION: (1) First notice that the series is convergent in norm and one can manipulate the power series as usual because any two powers of A commute with each other.

Next, check directly that the curve takes values on $\mathrm{O}(n)$: $\exp(tA)\exp(tA)^T = \exp(tA)\exp(tA^T) = \exp(tA)\exp(-tA) = I$, where we used $A^T = -A$. Finally

$$\frac{d}{dt}\exp(tA) = \exp(tA)\,A = A^\sharp_{\exp(tA)},$$

where the second equality is by the left-invariance of $A^\sharp$: (Left multiplication by $\exp(tA)$ is the restriction to $\mathrm{O}(n)$ of a linear map in the space of all $n\times n$ matrices and therefore its differential is again left multiplication.) This is exactly the condition that the curve be an integral curve of $A^\sharp$, and it clearly starts at the identity.

(2) For each $g \in \mathrm{O}(n)$, let $L_g : O(n) \to O(n)$ be $L_g(k) = gk$. Then left-invariance of $A^\sharp$ means that for each g $A^\sharp$ is L_g-related to itself. It follows that any integral curve of $A^\sharp$ followed by L_g is another integral curve of $A^\sharp$.

Therefore,

$$t \mapsto L_g(\exp(tA)) = g\exp(tA)$$

is an integral curve of $A^\sharp$, and it starts at g. Therefore $\phi_t(g) = g\exp(tA)$.

Problem 3.- Let $\mathcal{H}$ be the real vector space of all 2×2 complex Hermitian matrices. Let $0 < \lambda_1 < \lambda_2$ be two real numbers, and define

$$\mathcal{M} = \{A \in \mathcal{H}\ ;\ \text{the eigenvalues of } A \text{ are } \ \lambda_1, \lambda_2\}.$$

Show that $\mathcal{M}$ is a submanifold of $\mathcal{H}$. Find its dimension, and compute $T_D\mathcal{M}$ as a subspace of $\mathcal{H}$ where $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Hint: The eigenvalues of $A \in \mathcal{H}$ are determined by the trace and the determinant of A.

Solution: Introduce (real) linear coordinates (x_1, x_2, x_3, x_4) in $\mathcal{H}$ by letting

$$A = \begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_4 \end{pmatrix}.$$

In these coordinates, the map $F : \mathcal{H} \to \mathbb{R}^2$, $F(A) = (\mathrm{tr}A, \det A)$ is

$$F(x_1, x_2, x_3, x_4) = (x_3 + x_4, x_3x_4 - x_1^2 - x_2^2).$$

Note that $\mathcal{M}$ corresponds to $F^{-1}(\lambda_1 + \lambda_2, \lambda_1\lambda_2)$ in these coordinates, so we want to show that $(\lambda_1 + \lambda_2, \lambda_1\lambda_2)$ is a regular value of F in order to apply the regular value theorem. The Jacobian of F is

$$J = \begin{pmatrix} 0 & 0 & 1 & 1 \\ -2x_1 & -2x_2 & x_4 & x_3 \end{pmatrix}.$$

Assume $(x_3 + x_4, x_3x_4 - x_1^2 - x_2^2) = (\lambda_1 + \lambda_2, \lambda_1\lambda_2)$. If $x_3 \neq x_4$ the last two columns are linearly independent. If $x_3 = x_4$, then the trace condition implies $x_3x_4 = (\lambda_1 + \lambda_2)^2/4$, and the determinant condition implies $x_1^2 + x_2^2 = (\lambda_2 - \lambda_1)^2/4 > 0$, so at least one of x_1, x_2 is non-zero. In all cases J is full rank, and $\mathcal{M}$ is a manifold of dimension $4 - 2 = 2$.

At D, $J = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & \lambda_2 & \lambda_1 \end{pmatrix}$. Its kernel corresponds to the required tangent space, that is

$$T_D\mathcal{M} = \left\{ \begin{pmatrix} 0 & u_1 + iu_2 \\ u_1 - iu_2 & 0 \end{pmatrix}\ ;\ u_1, u_2 \in \mathbb{R} \right\}$$

Problem 4.- Let M be a manifold with boundary. A vector field $X \in \mathfrak{X}(M)$ is called a *b-field* iff

$$\forall p \in \partial M \qquad X_p \in T_p\partial M.$$

Show that the space of b-fields is closed under the Lie bracket.

SOLUTION: Let $(x_1, \dots x_n)$ be coordinates on $U \subset M$ intersecting the boundary, so that $U \cap \partial M = \{x_n = 0\}$. Let $X, Y \in \mathfrak{X}(M)$, and on U write them in coordinates as:

$$X|_U = \sum_{i=1}^{n} f_i \partial_i, \quad Y|_U = \sum_{i=1}^{n} g_i \partial_i$$

where $f_i, g_i \in C^\infty(U)$ and $\partial_i = \frac{\partial}{\partial x_i}$. The condition for X to be a b-field is that

$$f_n|_{x_n=0} = 0, \quad \text{and similarly for } Y. \tag{B}$$

Now a standard calculation yields

$$[X, Y]|_U = \sum_{ij} \left[f_i \frac{\partial g_j}{\partial x_i} - g_i \frac{\partial f_j}{\partial x_i} \right] \partial_j.$$

We are only interested in the ∂_n component, which is

$$\sum_{i=1}^{n} \left[f_i \frac{\partial g_n}{\partial x_i} - g_i \frac{\partial f_n}{\partial x_i} \right].$$

Assume X, Y are both b-fields and set $x_n = 0$. Taking into account (B), we see that for $i = 1, \dots, n-1$ the partial derivatives vanish. For $i = n$ f_n and g_n vanish. So all the terms in the sum above are zero when $x_n = 0$, which shows that $[X, Y]$ is a b-field.

Problem 5. Consider the unit sphere $S^n \subset \mathbb{R}^{n+1}$. Identify tangent spaces with subspaces of $\mathbb{R}^{n+1}$. Assume that there is a nowhere-vanishing smooth vector field X on S^n.

1. Show that the antipodal map $A : S^n \to S^n$, $A(p) = -p$ is homotopic to the identity. HINT: $\forall p \in S^n$, use X_p to define a great semi-circle connecting p and its antipode.

2. Show that n must be odd.

SOLUTION: (1) For each $p \in S^n$ let $V_p = \frac{1}{\|X_p\|} X_p$ (Euclidean norm). Then at each p $\{p, V_p\}$ are an orthonormal pair (thinking of p as a vector in $\mathbb{R}^{n+1}$). Define

$$F : [0, \pi] \times S^n \to S^n, \quad F(t, p) = \cos(t) p + \sin(t) V_p.$$

This is a smooth homotopy between the identity and the antipodal map.

(2) By the homotopy "axiom", the map $A^* : H^n(S^n) \to H^n(S^n)$ induced by the antipodal map in De Rham cohomology is the identity. Let $\nu \in \Omega^n(S^n)$ be the standard volume form. By a direct argument $A^* \nu = (-1)^{n-1} \nu$ (at the level of forms). Therefore $(-1)^{n-1} = 1$, i.e. n is odd.