Differentiable Manifolds QR Exam – January 5, 2021

All manifolds are assumed to be C^∞. All items will be graded independently of each other.

Problem 1.- Let $M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ ; \ x_1^2 + x_2^2 = 1 \text{ and } x_3^2 + x_4^2 = 1\}$, and let $\iota : M \hookrightarrow \mathbb{R}^4$ be the inclusion.

1. Show that M is a submanifold of $\mathbb{R}^4$.
2. If $\alpha = -x_2 dx_1 + x_1 dx_2 - x_4 dx_3 + x_3 dx_4$, show that $\iota^*(\alpha)$ is closed but not exact.

Problem 2.- Let $A \in T_I \mathrm{O}(n) \setminus \{0\}$ where I is the identity and $\mathrm{O}(n)$ the orthogonal group. Let $A^\sharp$ be the left-invariant vector field on $\mathrm{O}(n)$ whose value at the identity is A. Define $\forall t \in \mathbb{R}$

$$\exp(tA) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k.$$

1. Give a direct proof that $s \mapsto \exp(sA)$ is the integral curve of $A^\sharp$ starting at the identity.
2. Derive an expression of the time-t map $\phi_t : \mathrm{O}(n) \to \mathrm{O}(n)$ of the flow of $A^\sharp$ in terms of $\exp(tA)$.

Problem 3.- Let $\mathcal{H}$ be the real vector space of all 2×2 complex Hermitian matrices. Let $0 < \lambda_1 < \lambda_2$ be two real numbers, and define

$$\mathcal{M} = \{A \in \mathcal{H} \ ; \text{ the eigenvalues of } A \text{ are } \ \lambda_1, \lambda_2\}.$$

Show that $\mathcal{M}$ is a submanifold of $\mathcal{H}$. Find its dimension, and compute $T_D\mathcal{M}$ as a subspace of $\mathcal{H}$ where $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Hint: The eigenvalues of $A \in \mathcal{H}$ are determined by the trace and the determinant of A.

Problem 4.- Let M be a manifold with boundary. A vector field $X \in \mathfrak{X}(M)$ is called a *b-field* iff

$$\forall p \in \partial M \qquad X_p \in T_p \partial M.$$

Show that the space of b-fields is closed under the Lie bracket.

Problem 5. Consider the unit sphere $S^n \subset \mathbb{R}^{n+1}$. Identify tangent spaces with subspaces of $\mathbb{R}^{n+1}$. Assume that there is a nowhere-vanishing smooth vector field X on S^n.

1. Show that the antipodal map $A : S^n \to S^n$, $A(p) = -p$ is homotopic to the identity. Hint: $\forall p \in S^n$, use X_p to define a great semi-circle connecting x and its antipode.
2. Show that n must be odd.