Differential Topology QR Exam – Solutions
August 16, 2021

\( M \) denotes a \( C^\infty \) manifold of dimension \( n \).
\( \mathfrak{X}(M) \) is the space of all smooth vector fields on \( M \).
All items will be graded independently of each other, to the extent possible.

Problem 1.- Assume \( M \) is connected. Prove that between any two points there exists a smooth curve connecting them.

Solution: Define the relation on \( M \): \( p \sim q \) iff \( p \) and \( q \) can be joined by a smooth curve. Claim: this is an equivalence relation. The only non-trivial property is transitivity. Assume \( p \sim q \) and \( q \sim r \), and let \( \gamma_1 \) join \( p \) to \( q \) and \( \gamma_2 \) join \( q \) to \( r \). WLOG \( \gamma_1 : [a, b] \to M \) and \( \gamma_2 : [b, c] \to M \) with \( \gamma_1(a) = p, \gamma_1(b) = q = \gamma_2(b) \) and \( \gamma_2(c) = r \). Let \( \tilde{\gamma} : [a, c] \to M \) be the continuous piece-wise smooth curve agreeing with \( \gamma_1 \) on \([a, b]\) and with \( \gamma_2 \) on \([b, c]\). Using a smooth coordinate chart centered at \( q \), modify \( \tilde{\gamma} \) in a small neighborhood of \( b \) to yield a smooth curve \( \gamma : [a, c] \to M \) joining \( p \) and \( r \) (it is irrelevant if it does not pass by \( q \)). This proves transitivity.

Now pick \( p \in M \) and let \( \mathcal{C}_p \) be the equivalence class of \( p \). We claim that \( \mathcal{C}_p \) is open. Indeed let \( q \in \mathcal{C}_p \) and \( \phi : U \to \mathbb{R}^n \) a coordinate chart containing \( q \) with \( \phi(q) \) a Euclidean ball. Since any two points in a Euclidean ball can be joined by a smooth curve (say a straight line segment) the same is true for \( U \), and by transitivity \( U \subset \mathcal{C}_p \). Therefore the equivalence classes of \( \sim \) partition \( M \) into open sets, and since \( M \) is connected there is only one equivalence class.

Problem 2.- On vector fields:

1. Let \( X \in \mathfrak{X}(M) \), and assume that \( \exists \epsilon > 0 \) such that \( \forall p \in M \) the integral curve of \( X \) through \( p \) is defined \( \forall t \in (-\epsilon, \epsilon) \). Prove that \( X \) is complete.
2. Use (1) to show that every vector field on a compact manifold is complete.

Solution:

1. Arguing by contradiction, assume \( \exists p \in M \) such that the domain \( I \subset \mathbb{R} \) of the maximal integral curve \( \gamma_p : I \to M \) of \( X \) starting at \( p \) is bounded above, let \( \beta = \sup I \) be the upper endpoint of \( I \). Let \( T = \beta - \epsilon/2 \) and let \( q = \gamma_p(T) \). Define a curve \( \tilde{\gamma} : (-\epsilon, \beta + \epsilon/2) \to M \) by:

\[
\tilde{\gamma}(t) := \begin{cases} 
\gamma_p(t) & \text{if } t \in (-\epsilon, \beta) \\
\gamma_q(t - T) & \text{if } t \in (T, \beta + \epsilon/2)
\end{cases}
\]

where we use that the integral curve \( \gamma_q \) is defined on \(( -\epsilon, \epsilon) \). Both curves on the right-hand side of this definition are integral curves of \( X \) that agree on the overlap of their domains, by the group law of the flow \( \phi \) of \( X \):

\[
\gamma_q(t - T) = \phi_{t-T}(q) = \phi_{t-T}(\phi_T(p)) = \phi_t(p) = \gamma_p(t).
\]

Therefore (by uniqueness of integral curves) \( \tilde{\gamma} \) is an integral curve of \( X \) starting at \( p \). However \( \beta + \epsilon/2 > \beta \), which contradicts the definition of \( \beta \).

2. Let \( X \in \mathfrak{X}(M) \) with \( M \) compact. By the existence theorem of integral curves, \( \forall p \in M \exists U_p \) neighborhood of \( p \) and \( \exists \epsilon_p > 0 \) such that \( \forall q \in U_p \) the integral curve of \( X \) starting at \( p \) exists \( \forall t \in (-\epsilon_p, \epsilon_p) \). Let \( U_{p_1}, \ldots, U_{p_n} \) be a finite subcover of the cover \( \{U_p\}_{p \in M} \). Then \( \epsilon := \min\{\epsilon_{p_1}, \ldots, \epsilon_{p_n}\} \) and \( X \) satisfy the hypotheses of (1), and therefore \( X \) is complete.

Problem 3.- Two unrelated questions on Lie groups:
(1) Let \( U \subset G \) be a neighborhood of the identity where \( G \) is a Lie group. Show that there exists \( V \) a neighborhood of the identity such that \( V \subset U \) and \( \forall g, h \in V \ gh^{-1} \in U \).

(2) Show that every Lie group \( G \) is orientable, and that not all orientable manifolds admit a Lie group structure.

**Solution:**

(1) Consider the map \( F : G \times G \rightarrow G \) given by \( F(g, h) = gh^{-1} \). This is a smooth (and therefore continuous) map, and \( F(e, e) = e \) where \( e \) is the identity. By continuity, there exists a neighborhood \( W \subset G \times G \) of \( (e, e) \) such that \( F(W) \subset U \). By definition of the product topology, \( \exists V' \subset G \) a neighborhood of the identity such that \( V' \times V' \subset W \). Now let \( V = V' \cap U \).

(2) Let \( G \) be a Lie group of dimension \( n \) and \( \forall g \in G \), let \( L_g : G \rightarrow G \) be left-translation by \( g \), that is \( L_g(k) = gk \). Pick \( \nu_e \in \bigwedge^n g^* \setminus \{0\} \), and \( \forall g \in G \) define \( \nu_g = L_g^* \nu_e \in \bigwedge^n T_g^* M \). Then \( \nu \) is a smooth non vanishing top-degree form on \( G \), which shows that \( G \) is orientable.

OR: Pick \( e_1, \ldots, e_n \) a basis of \( g \) and extend them to left-invariant vector fields \( E_1, \ldots, E_n \) on \( G \). Evaluating these fields at each \( g \in G \) yields a basis of \( T_g G \). Define an orientation of \( G \) by declaring these basis to be positive.

For the converse, use that any Lie group has plenty of non-vanishing smooth vector fields: Any non-zero left-invariant field, for example. Such fields do not exist e.g. on \( S^2 \), which is orientable nonetheless.

**Problem 4.-** Let \( \alpha \) be a smooth one-form on \( M \). Assume that \( \forall p \in M \ \alpha_p \neq 0 \).

(1) Show that \( N := \{(p, v) \in TM \mid \alpha_p(v) = 0\} \) is a submanifold of the tangent bundle \( TM \).

(2) Assume that \( d\alpha = 0 \). Prove that \( \forall p \in M \) there exists a regular submanifold \( S \subset M \) such that \( p \in S \) and \( \forall q \in S \ T_q S = \ker \alpha_q \).

**Solution:**

(1) Let \( (x^1, \ldots, x^n) \) be coordinates on an open set \( U \subset M \), and \( (x^1, \ldots, x^n, v^1, \ldots, v^n) \) be the associated coordinates on \( TU \). Let \( \alpha = \sum_j a_j(x)dx^j \). Then \( N \cap TU \) is defined by the equation

\[
\sum_j a_j(x)v^j = 0,
\]

that is, \( N \cap TU \) is the zero level set of the function \( F(x, v) = \sum_j a_j(x)v^j \), \( F : TU \rightarrow \mathbb{R} \). The Jacobian of this function is

\[
(\nabla_x F, a_1(x), \ldots, a_n(x))^T
\]

which is nowhere zero since \( \alpha \) does not vanish. By the regular value theorem \( N \cap TU \) is a submanifold.

(2) Let \( p \in M \) and \( U \) a neighborhood of \( p \) diffeomorphic to a Euclidean ball. By the Poincaré lemma, \( \exists f \in C^\infty(U) \) such that \( \alpha|_U = df \). Since \( \alpha \) does not vanish, \( f \) has no critical points, and every \( c \in \mathbb{R} \) is a regular value of \( f \). Let \( S = f^{-1}(c) \), where \( c = f(p) \). Then \( S \) is a regular submanifold, \( p \in S \) and \( \forall q \in S \ T_q S = \ker df_q = \ker \alpha_q \).

**Problem 5.-** Let \( f : M \times W \rightarrow \mathbb{R} \) be a smooth function, where \( W \subset \mathbb{R}^k \) is open. For each \( (p, w) \in M \times W \), define the partial differential \( d_Mf_{(p,w)} \in T_p^* M \) by

\[
d_Mf_{(p,w)}(\gamma(0)) = \frac{d}{dt}f(\gamma(t), w)|_{t=0}
\]

for each smooth curve \( \gamma : (-\epsilon, \epsilon) \rightarrow M \) such that \( \gamma(0) = p \).
Assume that zero is a regular value of the map $\Phi : M \times W \to \mathbb{R}^k$ defined as

$$\Phi(p, w) = \left( \frac{\partial f}{\partial w^1}(p, w), \ldots, \frac{\partial f}{\partial w^k}(p, w) \right).$$

(1) Let $C := \Phi^{-1}(0)$. Explain why $C$ is a submanifold and compute its dimension. If $(p, x) \in C$ and $(x^1, \ldots, x^n)$ are coordinates in a neighborhood of $p$, write equations for the components $(\alpha^1, \ldots, \alpha^n, \beta^1, \ldots, \beta^k)$ of vectors in $T_{(p,w)}C$ in the coordinates $(x^1, \ldots, x^n, w^1, \ldots, w^k)$.

(2) Show that the map

$$F : C \to T^*M, \quad F(p, w) = (p, d_Mf_{(p,w)})$$

is an immersion.

Solution:

(1) The regular value theorem immediately implies that $C$ is a submanifold of $M \times W$, and that if $(p, w) \in C$ then

$$T_{(p,w)}C = \ker d\Phi_{(p,w)}.$$

By assumption the rank of $d\Phi_{(p,w)}$ is $k$, and therefore its kernel has dimension $n + k - k = n$, so $C$ has the same dimension as $M$.

Introduce coordinates $(x^1, \ldots, x^n)$ in a neighborhood of $p$. Then the matrix of $d\Phi_{(p,w)}$ (the Jacobian) is

$$J := \left( f_{wx} \quad f_{ww} \right),$$

where $f_{wx}$ is the matrix $f_{wx} = \left( \frac{\partial^2 f}{\partial x^i \partial w^j} \right)$, and similarly for $f_{ww}$. A tangent vector $\sum \alpha^i \partial x^i + \sum \beta^j \partial w^j$ is in $T_{(p,w)}C$ iff

$$f_{wx} \alpha + f_{ww} \beta = 0.$$

(2) Consider the extension $\tilde{F} : M \times W \to T^*M$ of $F$ given by the same expression as $F$. In coordinates,

$$\tilde{F}(x, w) = \left( x, \frac{\partial f}{\partial x^i}(x, w) \right),$$

and therefore the Jacobian matrix of $\tilde{F}$ is

$$K := \begin{pmatrix} I_n & 0 \\ f_{xx} & f_{xw} \end{pmatrix}.$$

Since $F = \tilde{F} \circ \iota$ where $\iota : F \hookrightarrow M \times W$ is the inclusion, the kernel of $dF_{(p,w)}$ is the space of vectors in $T_{(p,w)}C$ that are in the kernel of $\tilde{F}$. Therefore, the components of the vectors in $\ker dF_{(p,w)}$ are the intersection of the kernels of the matrices $J$ and $K$. Computing using column vectors $\alpha \in \mathbb{R}^n$, $\beta \in \mathbb{R}^k$:

$$J \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = f_{wx} \alpha + f_{ww} \beta, \quad K \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ f_{xx} \alpha + f_{xw} \beta \end{pmatrix},$$

we see that the joint kernel has for equations $\alpha = 0$, $f_{ww} \beta = 0$ and $f_{xw} \beta = 0$. Thus the equation on $\beta$ is

$$J^T \beta = 0.$$

Since $J$ is onto, $J^T$ is 1-1 and this equation implies that $\beta = 0$. Therefore $\ker dF_{(p,w)} = 0$, i.e. $F$ is an immersion.