Differential Topology QR Exam – Solutions
August 16, 2021

M denotes a C^∞ manifold of dimension n.
$\mathfrak{X}(M)$ is the space of all smooth vector fields on M.
All items will be graded independently of each other, to the extent possible.

Problem 1.- Assume M is connected. Prove that between any two points there exists a smooth curve connecting them.

Solution: Define the relation on M: $p \sim q$ iff p and q can be joined by a smooth curve. Claim: this is an equivalence relation. The only non-trivial property is transitivity. Assume $p \sim q$ and $q \sim r$, and let γ_1 join p to q and γ_2 join q to r. WOLOG $\gamma_1 : [a, b] \to M$ and $\gamma_2 : [b, c] \to M$ with $\gamma_1(a) = p$, $\gamma_1(b) = q = \gamma_2(b)$ and $\gamma_2(c) = r$. Let $\tilde{\gamma} : [a, c] \to M$ be the continuious piece-wise smooth curve agreeing with γ_1 on $[a, b]$ and with γ_2 on $[b, c]$. Using a smooth coordinate chart centered at q, modify $\tilde{\gamma}$ in a small neighborhood of b to yield a smooth curve $\gamma : [a, c] \to M$ joining p and r (it is irrelevant if it does not pass by q). This proves transitivity.

Now pick $p \in M$ and let $\mathcal{C}_p$ be the equivalence class of p. We claim that $\mathcal{C}_p$ is open. Indeed let $q \in \mathcal{C}_p$ and $\phi : U \to \mathbb{R}^n$ a coordinate chart containing q with $\phi(U)$ a Euclidean ball. Since any two points in a Euclidean ball can be joined by a smooth curve (say a straight line segment) the same is true for U, and by transitivity $U \subset \mathcal{C}_p$. Therefore the equivalence classes of $\sim$ partition M into open sets, and since M is connected there is only one equivalence class.

Problem 2.- On vector fields:

(1) Let $X \in \mathfrak{X}(M)$, and assume that $\exists \epsilon > 0$ such that $\forall p \in M$ the integral curve of X through p is defined $\forall t \in (-\epsilon, \epsilon)$. Prove that X is complete.
(2) Use (1) to show that every vector field on a compact manifold is complete.

Solution:

(1) Arguing by contradiction, assume $\exists p \in M$ such that the domain $I \subset \mathbb{R}$ of the maximal integral curve $\gamma_p : I \to M$ of X starting at p is bounded above, let $\beta = \sup I$ be the upper endpoint of I. Let $T = \beta - \epsilon/2$ and let $q = \gamma_p(T)$. Define a curve $\tilde{\gamma} : (-\epsilon, \beta + \epsilon/2) \to M$ by:

$$\tilde{\gamma}(t) := \begin{cases} \gamma_p(t) & \text{if } t \in (-\epsilon, \beta) \\ \gamma_q(t - T) & \text{if } t \in (T, \beta + \epsilon/2) \end{cases}$$

where we use that the integral curve γ_q is defined on $(-\epsilon, \epsilon)$. Both curves on the right-hand side of this definition are integral curves of X that agree on the overlap of their domains, by the group law of the flow ϕ of X:

$$\gamma_q(t - T) = \phi_{t-T}(q) = \phi_{t-T}(\phi_T(p)) = \phi_t(p) = \gamma_p(t).$$

Therefore (by uniqueness of integral curves) $\tilde{\gamma}$ is an integral curve of X starting at p. However $\beta + \epsilon/2 > \beta$, which contradicts the definition of β.

(2) Let $X \in \mathfrak{X}(M)$ with M compact. By the existence theorem of integral curves, $\forall p \in M$ $\exists U_p$ neighborhood of p and $\exists \epsilon_p > 0$ such that $\forall q \in U_p$ the integral curve of X starting at p exists $\forall t \in (-\epsilon_p, \epsilon_p)$. Let $U_{p_1}, \ldots, U_{p_N}$ be a finite subcover of the cover $\{U_p\}_{p \in M}$. Then $\epsilon := \min\{\epsilon_{p_1}, \ldots, \epsilon_{p_N}\}$ and X satisfy the hypothses of (1), and therefore X is complete.

Problem 3.- Two unrelated questions on Lie groups:

(1) Let $U \subset G$ be a neighborhood of the identity where G is a Lie group. Show that there exists V a neighborhood of the identity such that $V \subset U$ and $\forall g, h \in V$ $gh^{-1} \in U$.
(2) Show that every Lie group G is orientable, and that not all orientable manifolds admit a Lie group structure.

Solution:

(1) Consider the map $F : G \times G \to G$ given by $F(g,h) = gh^{-1}$. This is a smooth (and therefore continuous) map, and $F(e,e) = e$ where e is the identity. By continuity, there exists a neighborhood $W \subset G \times G$ of (e,e) such that $F(W) \subset U$. By definition of the product topology, $\exists V' \subset G$ a neighborhood of the identity such that $V' \times V' \subset W$. Now let $V = V' \cap U$.

(2) Let G be a Lie group of dimension n and $\forall g \in G$, let $L_g : G \to G$ be left-translation by g, that is $L_g(k) = gk$. Pick $\nu_e \in \bigwedge^n \mathfrak{g}^* \setminus \{0\}$, and $\forall g \in G$ define $\nu_g = L^*_{g^{-1}} \nu_e \in \bigwedge^n T^*_g M$. Then ν is a smooth non vanishing top-degree form on G, which shows that G is orientable.

OR: Pick $e_1, \ldots, e_n$ a basis of $\mathfrak{g}$ and extend them to left-invariant vector fields $E_1, \ldots, E_n$ on G. Evaluating these fields at each $g \in G$ yields a basis of $T_g G$. Define an orientation of G by declaring these basis to be positive.

For the converse, use that any Lie group has plenty of non-vanishing smooth vector fields: Any non-zero left-invariant field, for example. Such fields do not exist e.g. on S^2, which is orientable nonetheless.

Problem 4.- Let α be a smooth one-form on M. Assume that $\forall p \in M$ $\alpha_p \neq 0$.

(1) Show that $\mathcal{N} := \{(p,v) \in TM \mid \alpha_p(v) = 0\}$ is a submanifold of the tangent bundle TM.
(2) Assume that $d\alpha = 0$. Prove that $\forall p \in M$ there exists a regular submanifold $S \subset M$ such that $p \in S$ and $\forall q \in S$ $T_q S = \ker \alpha_q$.

Solution:

(1) Let $(x^1, \ldots, x^n)$ be coordinates on an open set $U \subset M$, and $(x^1, \ldots, x^n, v^1, \ldots v^n)$ be the associated coordinates on TU. Let $\alpha = \sum_j a_j(x) dx^j$. Then $\mathcal{N} \cap TU$ is defined by the equation

$$\sum_j a_j(x) v^j = 0,$$

that is, $\mathcal{N} \cap TU$ is the zero level set of the function $F(x,v) = \sum_j a_j(x) v^j$, $F : TU \to \mathbb{R}$. The Jacobian of this function is

$$(\nabla_x F, a_1(x), \ldots, a_n(x))^T$$

which is nowhere zero since α does not vanish. By the regular value theorem $\mathcal{N} \cap TU$ is a submanifold.

(2) Let $p \in M$ and U a neighborhood of p diffeomorphic to a Euclidean ball. By the Poincaré lemma, $\exists f \in C^\infty(U)$ such that $\alpha|_U = df$. Since α does not vanish, f has no critical points, and every $c \in \mathbb{R}$ is a regular value of f. Let $S = f^{-1}(c)$, where $c = f(p)$. Then S is a regular submanifold, $p \in S$ and $\forall q \in S$ $T_q S = \ker df_q = \ker \alpha_q$.

Problem 5.- Let $f : M \times W \to \mathbb{R}$ be a smooth function, where $W \subset \mathbb{R}^k$ is open. For each $(p,w) \in M \times W$, define the partial differential $d_M f_{(p,w)} \in T^*_p M$ by

$$d_M f_{(p,w)}(\dot\gamma(0)) = \frac{d}{dt} f(\gamma(t), w)|_{t=0}$$

for each smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$.

Assume that zero is a regular value of the map $\Phi : M \times W \to \mathbb{R}^k$ defined as

$$\Phi(p, w) = \left(\frac{\partial f}{\partial w^1}(p, w), \ldots, \frac{\partial f}{\partial w^k}(p, w) \right).$$

(1) Let $\mathcal{C} := \Phi^{-1}(0)$. Explain why $\mathcal{C}$ is a submanifold and compute its dimension. If $(p, x) \in \mathcal{C}$ and $(x^1, \ldots, x^n)$ are coordinates in a neighborhood of p, write equations for the components $(\alpha^1, \ldots, \alpha^n, \beta^1, \ldots, \beta^k)$ of vectors in $T_{(p,w)}\mathcal{C}$ in the coordinates $(x^1, \ldots, x^n, w^1, \ldots, w^k)$.

(2) Show that the map

$$F : \mathcal{C} \to T^*M, \qquad F(p, w) = \big(p, d_M f_{(p,w)}\big)$$

is an immersion.

Solution:

(1) The regular value theorem immediately implies that $\mathcal{C}$ is a submanifold of $M \times W$, and that if $(p, w) \in \mathcal{C}$ then

$$T_{(p,w)}\mathcal{C} = \ker d\Phi_{(p,w)}.$$

By assumption the rank of $d\Phi_{(p,w)}$ is k, and therefore its kernel has dimension $n + k - k = n$, so $\mathcal{C}$ has the same dimension as M.

Introduce coordinates $(x^1, \ldots, x^n)$ in a neighborhood of p. Then the matrix of $d\Phi_{(p,w)}$ (the Jacobian) is

$$J := \begin{pmatrix} f_{wx} & f_{ww} \end{pmatrix},$$

where f_{wx} is the matrix $f_{wx} = \left(\frac{\partial^2 f}{\partial x^i w^j} \right)$, and similarly for f_{ww}. A tangent vector $\sum \alpha^i \partial_{x^i} + \sum \beta^j \partial_{w^j}$ is in $T_{(p,w)}\mathcal{C}$ iff

$$f_{wx}\alpha + f_{ww}\beta = 0.$$

(2) Consider the extension $\widetilde{F} : M \times W \to T^*M$ of F given by the same expression as F. In coordinates,

$$\widetilde{F}(x, w) = \left(x, \frac{\partial f}{\partial x^i}(x, w) \right),$$

and therefore the Jacobian matrix of $\widetilde{F}$ is

$$K := \begin{pmatrix} I_n & 0 \\ f_{xx} & f_{xw} \end{pmatrix}.$$

Since $F = \widetilde{F} \circ \iota$ where $\iota : F \hookrightarrow M \times W$ is the inclusion, the kernel of $dF_{(p,w)}$ is the space of vectors in $T_{(p,w)}\mathcal{C}$ that are in the kernel of $\widetilde{F}$. Therefore, the components of the vectors in $\ker dF_{(p,w)}$ are the intersection of the kernels of the matrices J and K. Computing using column vectors $\alpha \in \mathbb{R}^n$, $\beta \in \mathbb{R}^k$:

$$J \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = f_{wx}\alpha + f_{ww}\beta, \quad K \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ f_{xx}\alpha + f_{xw}\beta, \end{pmatrix}$$

we see that the joint kernel has for equations $\alpha = 0$, $f_{ww}\beta = 0$ and $f_{xw}\beta = 0$. Thus the equation on β is

$$J^T\beta = 0.$$

Since J is onto, J^T is 1-1 and this equation implies that $\beta = 0$. Therefore $\ker dF_{(p,w)} = 0$, i.e. F is an immersion.