Algebraic Topology QR Exam – August 2021

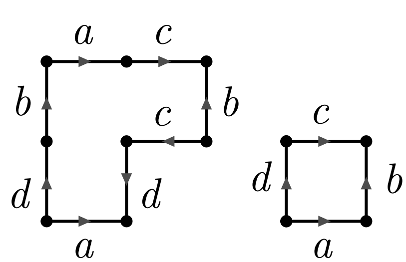
1. Let $n \geq 0$. Let $\mathbb{C}\mathrm{P}^n$ denote complex projective n-space, and let $x_0 \in \mathbb{C}\mathrm{P}^n$ be a fixed basepoint. Let S^1 denote the circle, and let $y_0 \in S^1$ be a fixed basepoint. Give an explicit proof that every based map
$$f : (\mathbb{C}\mathrm{P}^n, x_0) \to (S^1, y_0)$$
is nullhomotopic via a *basepoint-preserving homotopy*, i.e., a homotopy f_t satisfying $f_t(x_0) = y_0$ for all t.

2. Let F_n denote the free group on n letters $\{a, b, c, \ldots\}$.
   (a) Prove that F_4 does **not** have a finite-index subgroup isomorphic to F_8.
   (b) Construct a finite-index subgroup H of F_4 isomorphic to F_7. Determine (explaining your steps) a free generating set for H, and explain whether H is normal.

3. Fix $n \geq 1$. Let S^n denote the n-sphere, and let $f : S^n \to S^n$ be a (non-identity) deck transformation associated to a certain covering space map $S^n \to X$. What can you say about the degree of f as a map $S^n \to S^n$?

4. Fix $d \geq 1$. Let X denote a d-dimensional Δ-complex, and suppose that X is homotopy equivalent to a d-sphere. Let Y denote the $(d-1)$-skeleton of X. Prove that
$$\widetilde{H}_i(Y) = 0 \qquad \text{for } i \neq d-1$$
and $\widetilde{H}_{d-1}(Y)$ is generated by cycles equal to the boundaries of d-simplices of X,
$$\{\partial\Delta_i \mid \Delta_i \text{ a } d\text{-simplex of } X\} \subseteq C_{d-1}(Y).$$

5. A space X is constructed from two polygons with the following edge identifications. Compute the homology of X.



Solutions

These solutions may include more detail than is necessary for full points.

1. Let $p : \mathbb{R} \to S^1$ be the universal covering map, and let $r_0 \in p^{-1}(y_0)$ be any choice of preimage of the basepoint. We will invoke the *lifting criterion* for covering spaces,

   Theorem. Suppose $p : (\tilde{Y}, r_0) \to (Y, y_0)$ is a covering space map, and suppose $f : (X, x_0) \to (Y, y_0)$ a based map with X path-connected and locally path-connected. Then a lift $\tilde{f} : (X, x_0) \to (\tilde{Y}, r_0)$ of f exists if and only if $f_*(\pi_1(X, x_0)) \subseteq p_*(\pi_1(\tilde{Y}, r_0))$.

   Since $\mathbb{C}\mathrm{P}^n$ is a connected CW complex, it is path-connected and locally path-connected. Since $\pi_1(\mathbb{C}\mathrm{P}^n) = 0$ for all n, the condition on fundamental groups is vacuously satisfied. Thus we have a map
$$\tilde{f} : (\mathbb{C}\mathrm{P}^n, x_0) \to (\mathbb{R}, r_0)$$
   such that $p \circ \tilde{f} = f$. Let $h_t : \mathbb{R} \to \mathbb{R}$ be straight-line homotopy from $id_{\mathbb{R}}$ to the constant map at r_0,
$$h_t(r) = r_0 + (1 - t)(r - r_0)$$
   Notably, $h_t(r_0) = r_0$ for all t. Then define
$$f_t(x) = p \circ h_t \circ \tilde{f}(x).$$
$$\begin{tikzcd} & (\mathbb{R}, r_0) \arrow[d, "p"] \arrow[loop right, "h_t"] \\ (\mathbb{C}\mathrm{P}^n, x_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & (S^1, y_0) \end{tikzcd}$$
   Then when $t = 0$,
$$f_0(x) = p \circ h_0 \circ \tilde{f}(x) = p \circ id_{\mathbb{R}} \circ \tilde{f}(x) = f(x) \qquad \text{for all } x.$$
   When $t = 1$,
$$f_1(x) = p \circ h_1(\tilde{f}(x)) = p(r_0) = y_0 \qquad \text{for all } x.$$
   When $x = x_0$,
$$f_t(x_0) = p \circ h_t \circ \tilde{f}(x_0) = p \circ h_t(r_0) = p(r_0) = y_0 \qquad \text{for all } t.$$
   Thus f_t is a basepoint-preserving nullhomotopy as desired.

2(a). We will use covering space theory to show that every finite-index subgroup of F_4 must be a free group with rank congruent to 1 mod 3. In particular, F_8 is not a finite-index subgroup of F_4.

We can realize F_4 as the fundamental group of the wedge
$$X = S^1 \vee S^1 \vee S^1 \vee S^1.$$
The classification of covering spaces then states that any index-d subgroup H of F_4 must be the fundamental group of some degree-d cover $\tilde{X}$ of X.

Approach one: Since X is a graph with 1 vertex and 4 edges, the cover $\tilde{X}$ has the structure of a graph with d vertices and $4d$ edges. The graph $\tilde{X}$ is homotopy equivalent to a wedge of circles, via the quotient collapsing a maximal tree. A tree with d vertices has $(d-1)$ edges. Hence, $\tilde{X}$ is homotopy equivalent to a wedge of

$$4d - (d-1) = 3d + 1$$

circles, and $\pi_1(\tilde{X})$ is a free group of rank $3d+1$.

Approach two: Recall that the Euler characteristic of a finite graph G is

$$\chi(G) = \#\text{vertices} - \#\text{edges}.$$

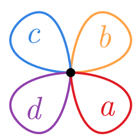
A wedge W of n circles has fundamental group $\pi_1(W) \cong F_n$ and Euler characteristic $\chi(W) = 1 - n$. Thus $\pi_1(W) \cong F_{-\chi(W)+1}$. Fundamental group and Euler characteristic are homotopy invariants, and every graph is homotopy equivalent to a wedge of circles, so the formula

$$\pi_1(G) \cong F_{-\chi(G)+1}$$

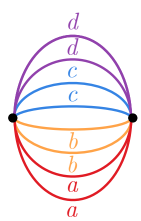
holds for every finite graph G. In particular, $\chi(\tilde{X}) = d\chi(X) = -3d$, and

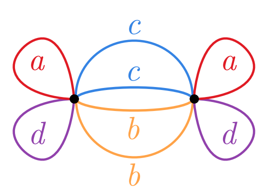
$$\pi_1(\tilde{X}) \cong F_{3d+1}.$$

2(b). Let $X = S^1 \vee S^1 \vee S^1 \vee S^1$.



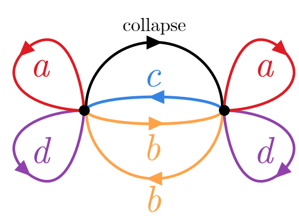
Any connected degree-2 cover $\tilde{X}$ of X will have fundamental group F_7. The data of a degree-2 cover is a directed graph with 8 edges and 2 vertices, such that each vertex has in-degree 4 and out-degree 4. Two edges should be labelled by each of a, b, c, d such that each vertex has one incoming and one outgoing edge with each label. Some examples are given below (orientations not shown).





To find a generating set for the fundamental group: choose a maximal tree T (in this case, a single edge spanning the two vertices) to collapse. The quotient is a wedge of 7 circles and the quotient map is a homotopy equivalence. Choose a vertex in $\tilde{X}$ to be the basepoint. Choose a based preimage in $\tilde{X}$ of each of the 7 circles in $\tilde{X}/T$; these loops freely generate $\pi_1(\tilde{X})$. The image of these generators in X are free generators for H.

For example, if we choose the following cover and maximal tree, and left vertex as basepoint,



we obtain free generating set

$$H = \langle a, d, c^2, cb, cb^{-1}, cac^{-1}, cdc^{-1} \rangle.$$

Since H is the image of the fundamental group of a connected degree-2 cover, it has index 2 in F_4, and hence is necessarily normal. We can also see this by verifying that the cover is a regular cover: there is a deck transformation of the cover (a graph automorphism preserving directed, labelled edges) that interchanges the two vertices. Thus the deck group acts transitively on fibers, the cover is regular, and the image of its fundamental group is normal in $\pi_1(X)$.

3. We will show that the degree of f is $(-1)^{n+1}$.

   First we claim that, since f is a non-identity deck transformation, it has no fixed points. Recall that, if $p : \tilde{X} \to X$ is a covering space map, then a *deck transformation* is a homeomorphism $f : \tilde{X} \to \tilde{X}$ such that $p \circ f = p$.

   Recall that covering spaces satisfy the *unique lifting property*:

   > **Proposition.** Given a covering space $p : \tilde{X} \to X$ and a map $g : Y \to X$, if two lifts $f_1, f_2 : Y \to X$ of g agree at one point of Y and Y is connected, then f_1 and f_2 agree on all of Y.

   If we take $g : Y \to X$ to again be the covering map $p : \tilde{X} \to X$, then a deck map $f : \tilde{X} \to \tilde{X}$ is a lift of p, and the unique lifting property implies that f is determined by its value at any single point. In particular, if f fixes a point then it must be the identity map.

   So we conclude that our non-identity deck map f has no fixed points. This allows us to compute its degree.

   Approach one: Lefschetz fixed point theorem. Because the sphere S^n has the structure of a compact manifold (and, in fact, a finite CW complex), the Lefschetz Fixed Point Theorem applies to maps $S^n \to S^n$. Because our map $f : S^n \to S^n$ has no fixed points, the theorem states that it must have Lefschetz number 0,

   $$0 = \sum_{i=0}^{\infty} (-1)^i \operatorname{Trace}\Big(f_* : H_i(S^n) \to H_i(S^n) \Big).$$

   The space S^n has nonzero homology groups only in degrees 0 and n, and by assumption $n \neq 0$. Hence

   $$0 = \operatorname{Trace}\Big(f_* : H_0(S^n) \to H_0(S^n) \Big) + (-1)^n \operatorname{Trace}\Big(f_* : H_n(S^n) \to H_n(S^n) \Big)$$

   Since $n \geq 1$, the sphere S^n is path-connected. Thus $H_0(S^n) \cong \mathbb{Z}$ and the map f must induce the identity map on $H_0(S^n)$. Then $\operatorname{Trace}\Big(f_* : H_0(S^n) \to H_0(S^n) \Big) = 1$, and

   $$\begin{aligned} 0 &= 1 + (-1)^n \operatorname{Trace}\Big(f_* : H_n(S^n) \to H_n(S^n) \Big) \\ -1 &= (-1)^n \operatorname{Trace}\Big(f_* : H_n(S^n) \to H_n(S^n) \Big) \\ (-1)^{n+1} &= \operatorname{Trace}\Big(f_* : H_n(S^n) \to H_n(S^n) \Big) \end{aligned}$$

   The induced map f_* on the rank-one group $H_n(S^n) \cong \mathbb{Z}$ can be represented by a 1×1 matrix; the matrix's single entry is both the trace of f_* and (by definition) the degree of the map f. We conclude that f has degree $(-1)^{n+1}$.

Approach two: homotopy to the antipodal map. Since f has no fixed points, we will show that it is homotopic to the antipodal map A. View S^n as the unit sphere in $\mathbb{R}^{n+1}$, so A is defined to be the map

$$\begin{aligned} A : S^n &\longrightarrow S^n \\ x &\longmapsto -x. \end{aligned}$$

Since f has no fixed points, the denominator in the following formula is non-vanishing, and the formula gives a continuous homotopy from f to A.

$$F_t(x) = \frac{(1-t)f(x) - tx}{||(1-t)f(x) - tx||}$$

But, the antipodal map can be written as a product of $(n+1)$ reflections, which (by a direct calculation) each have degree -1. Since degree is homotopy invariant, we conclude that f has degree $(-1)^{n+1}$.

4. *Approach one: the long exact sequence of a pair.* Since Y is a closed subcomplex of the Δ-complex X, the pair (X, Y) is what Hatcher calls a *good pair*, and the relative homology groups $H_*(X, Y)$ are equal to the reduced homology of the quotient $\widetilde{H}_*(X/Y)$.
   Because X is a Δ-complex of dimension d and Y is its $(d-1)$-skeleton, the quotient X/Y is homotopy equivalent to a wedge of d-spheres, with one d-sphere given by the image of each d-simplex Δ_i of X. The simplices $\Delta_i \in C_d(X, Y)$ are themselves relative cycles that freely generate the degree-d homology group.

$$H_i(X,Y) = \widetilde{H}_i(X/Y) = \begin{cases} 0, & i \neq d \\ \bigoplus_{\Delta_i \text{ a } d\text{-simplex of } X} \mathbb{Z}\{\Delta_i\}, & i = d. \end{cases}$$

   [This description of $H_*(X, Y)$ is a standard result that is used, for example, in the proof that cellular and singular homology agrees.]
   Since X is homotopy equivalent to a d-sphere,

$$\widetilde{H}_i(X) = 0 \quad \text{for } i \neq d.$$

   Then, for $i \leq d-2$, the long exact sequence of a pair

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H_{i+1}(X,Y) & \xrightarrow{\partial} & \widetilde{H}_i(Y) & \longrightarrow & \widetilde{H}_i(X) & \longrightarrow & H_i(X,Y) & \xrightarrow{\partial} \cdots \\ & \| & & & & \| & & & \\ & 0 & & & & 0 & & & \end{array}$$

   implies that $\widetilde{H}_i(Y) = 0$ for $i \leq d-2$. When $i \geq d$ then $\widetilde{H}_i(Y) = 0$ because Y is a $(d-1)$-dimensional complex.
   When $i = d-1$,

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H_d(X,Y) & \xrightarrow{\partial} & \widetilde{H}_{d-1}(Y) & \longrightarrow & \widetilde{H}_{d-1}(X) & \longrightarrow & H_{d-1}(X,Y) & \xrightarrow{\partial} \cdots \\ & \| & & & & \| & & & \\ & \bigoplus_{\Delta_i \text{ a } d\text{-simplex of } X} \mathbb{Z}\{\Delta_i\} & & & & 0 & & & \end{array}$$

by exactness the connecting homomorphism ∂ surjects, and $\widetilde{H}_{d-1}(Y)$ is equal to the image of

$$H_d(X,Y) = \bigoplus_{\Delta_i \text{ a } d\text{-simplex of } X} \mathbb{Z}\{\Delta_i\}$$

under ∂. But the connecting homomorphism acts by taking a relative cycle $\alpha \in C_i(X,Y)$ to its boundary $\partial\alpha \in C_{i-1}(Y)$, so in this case the image of $\widetilde{H}_{d-1}(Y)$ is generated by the cycles

$$\{\partial\Delta_i \mid \Delta_i \text{ a } d\text{-simplex of } X\},$$

as claimed.

Approach two: direct analysis of the chain complexes. View the augmented simplicial chain complex $C_*(Y)$ as a sub-chain complex of the augmented simplicial chain complex $C_*(X)$.

$$\begin{tikzcd}
0 \arrow[r] & C_d(X) \arrow[r, "\partial_d^X"] & C_{d-1}(X) \arrow[r, "\partial_{d-1}^X"] \arrow[d, equal] & C_{d-2}(X) \arrow[r, "\partial_{d-2}^X"] \arrow[d, equal] & \cdots \arrow[r, "\partial_2^X"] & C_1(X) \arrow[r, "\partial_1^X"] \arrow[d, equal] & C_0(X) \arrow[r] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & 0 \arrow[r, "\partial_d^Y"] & C_{d-1}(Y) \arrow[r, "\partial_{d-1}^Y"] & C_{d-2}(Y) \arrow[r, "\partial_{d-2}^Y"] & \cdots \arrow[r, "\partial_2^Y"] & C_1(Y) \arrow[r, "\partial_1^Y"] & C_0(Y) \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

For $i \leq d-2$, we have equality of maps (including equality of domains and codomains) $\partial_{i+1}^X = \partial_{i+1}^Y$ and $\partial_i^X = \partial_i^Y$,

$$\begin{tikzcd}
C_{i+1}(X) \arrow[r, "\partial_{i+1}^X"] \arrow[d, equal] & C_i(X) \arrow[r, "\partial_i^X"] \arrow[d, equal] & C_{i-1}(X) \arrow[d, equal] \\
C_{i+1}(Y) \arrow[r, "\partial_{i+1}^Y"] & C_i(Y) \arrow[r, "\partial_i^Y"] & C_{i-1}(Y)
\end{tikzcd}$$

so $\widetilde{H}_i(Y) = \widetilde{H}_i(X)$ for $i \leq d-2$. Since X is homotopy equivalent to a d-sphere, these groups vanish.

When $i \geq d > \dim(Y)$, the simplicial chain groups $C_i(Y)$ vanish and $\widetilde{H}_i(Y) = 0$.

Let $i = d-1$. Since X is homotopy equivalent to a d-sphere, $\widetilde{H}_{d-1}(X) = 0$ and the simplicial chain complex $C_*(X)$ is exact at $C_{d-1}(X)$. This means that the kernel of ∂_{d-1}^X is equal to the image of ∂_d^X. By definition of the boundary map this image of ∂_d^X is generated by $\{\partial\Delta_i \mid \Delta_i \text{ a } d\text{-simplex of } X\}$.

Since $C_d(Y) = 0$, the homology of the chain complex $C_*(Y)$ at $C_{d-1}(Y)$ is equal to

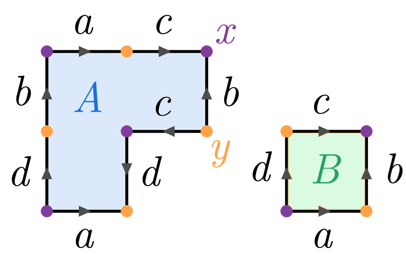
$$\ker(\partial_{d-1}^Y) = \ker(\partial_{d-1}^X),$$

and we conclude that $\widetilde{H}_{d-1}(Y)$ is the submodule of $C_{d-1}(Y) = C_{d-1}(X)$ spanned by $\{\partial\Delta_i \mid \Delta_i \text{ a } d\text{-simplex of } X\}$.

5. The solution is:

$$\begin{aligned}
H_0(X) &\cong \mathbb{Z} \\
H_1(X) &\cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \\
H_2(X) &\cong 0 \\
H_i(X) &\cong 0 \qquad \text{for all } i \geq 3.
\end{aligned}$$

The space X is a CW complex with two vertices x, y, four 1-cells a, b, c, d, and two 2-cells A, B.



Its cellular chain complex is:

$$0 \longrightarrow C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \longrightarrow 0$$

$$A \longmapsto a + c - b + c + d - a + d + b = 2c + 2d$$

$$B \longmapsto d + c - b - a$$

$$a \longmapsto y - x$$

$$b \longmapsto x - y$$

$$c \longmapsto x - y$$

$$d \longmapsto y - x$$

It is possible to compute the homology either by direct algebraic manipulation, or by using row/column operations to put the matrices

$$\partial_2 = \begin{bmatrix} 0 & -1 \\ 0 & -1 \\ 2 & 1 \\ 2 & 1 \end{bmatrix} \qquad \partial_1 = \begin{bmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

into Smith normal form.

Approach one: direct manipulation. The space X is path-connected, so $H_0(X) \cong \mathbb{Z}$. The map ∂_1 has rank 1 and so its kernel is a free abelian group of rank 3. By inspection, the set $\{a, a+b, b-c, c+d\}$ is a $\mathbb{Z}$-basis for $C_1(X)$ with $\{a+b, b-c, c+d\} \subseteq \ker(\partial_1)$, so $\{a+b, b-c, c+d\}$ must be a basis for the kernel. Quotienting by image of ∂_2 imposes the relations $2(c+d) = 0$ and $(c+d) = (a+b)$, and we conclude that the first homology group is

$$H_1(X) = \frac{\mathbb{Z}\{a+b, b-c, c+d\}}{\langle 2(c+d), (c+d) - (a+b) \rangle} \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Observe that $\partial_2(mA + nB) = (2m+n)d + (2m+n)c - nb - na$, $(m, n \in \mathbb{Z})$, which is zero in $C_1(X)$ only if $m = n = 0$. We conclude that ∂_2 is injective and $H_2(X) = 0$.

Approach two: Smith Normal Form.

$$\mathrm{SNF}(\partial_2) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathrm{SNF}(\partial_1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The solution follows frorm the formula

$$H_i(X) \cong \mathbb{Z}^{\text{rank}(C_i(X))-\text{rank}(\partial_i)-\text{rank}(\partial_{i+1})} \oplus \bigoplus_{\text{invariant factors } \alpha_k \text{ of } \partial_{i+1}} \mathbb{Z}/\alpha_k\mathbb{Z}.$$

Note: X is **not** a surface (it has triples of edges glued together so it is not locally Euclidean), and we cannot compute its homology using Euler characteristic / classification of surfaces arguments.