Algebraic Topology QR Exam – August 2021

1. Let $n \geq 0$. Let $\mathbb{C}\mathrm{P}^n$ denote complex projective n-space, and let $x_0 \in \mathbb{C}\mathrm{P}^n$ be a fixed basepoint. Let S^1 denote the circle, and let $y_0 \in S^1$ be a fixed basepoint. Give an explicit proof that every based map
$$f : (\mathbb{C}\mathrm{P}^n, x_0) \to (S^1, y_0)$$
is nullhomotopic via a *basepoint-preserving homotopy*, i.e., a homotopy f_t satisfying $f_t(x_0) = y_0$ for all t.

2. Let F_n denote the free group on n letters $\{a, b, c, \ldots\}$.
   (a) Prove that F_4 does **not** have a finite-index subgroup isomorphic to F_8.
   (b) Construct a finite-index subgroup H of F_4 isomorphic to F_7. Determine (explaining your steps) a free generating set for H, and explain whether H is normal.

3. Fix $n \geq 1$. Let S^n denote the n-sphere, and let $f : S^n \to S^n$ be a (non-identity) deck transformation associated to a certain covering space map $S^n \to X$. What can you say about the degree of f as a map $S^n \to S^n$?

4. Fix $d \geq 1$. Let X denote a d-dimensional Δ-complex, and suppose that X is homotopy equivalent to a d-sphere. Let Y denote the $(d-1)$-skeleton of X. Prove that
$$\widetilde{H}_i(Y) = 0 \qquad \text{for } i \neq d-1$$
and $\widetilde{H}_{d-1}(Y)$ is generated by cycles equal to the boundaries of d-simplices of X,
$$\{\partial\Delta_i \mid \Delta_i \text{ a } d\text{-simplex of } X\} \subseteq C_{d-1}(Y).$$

5. A space X is constructed from two polygons with the following edge identifications. Compute the homology of X.