Differential Topology QR Exam – August 22, 2020

All manifolds are assumed to be C^∞

All items will be graded independently of each other, to the extent possible.

Problem 1.- Consider the action of the multiplicative group $\mathbb{R}^+$ of positive real numbers on $X = \mathbb{R}^2 \setminus \{0\}$ given by:

$$\forall \lambda \in \mathbb{R}^+,\ x = (x_1, x_2) \in X \qquad \lambda \cdot x = (\lambda x_1, \lambda^{-1} x_2).$$

(1) Show that the orbit space $X/\mathbb{R}^+$, with the quotient topology, is not Hausdorff.

(2) Let $S = \{(x_1, x_2) \in X \ ;\ x_1 \neq 0 \text{ and } x_2 \neq 0\}$. Note that the action of $\mathbb{R}^+$ on X preserves S. Prove that $S/\mathbb{R}^+$ is Hausdorff.

Problem 2.- Let $F = (f^1, \ldots, f^k) : M \to \mathbb{R}^k$ be a smooth map, and assume that zero is a regular value. Let $S = F^{-1}(0)$, and let $\iota : S \hookrightarrow M$ be the inclusion. Let $\phi : M \to \mathbb{R}$ smooth, and $g := \phi \circ \iota$.

Prove that a point $p \in S$ is a critical point of g iff $d\phi_p$ is in the span of $\{df^1_p, \ldots, df^k_p\}$.

Problem 3.- Let $M = \mathbb{R}^2/\mathbb{Z}^2$ with the natural smooth manifold structure. Let $\gamma : [0,1] \to M$ be given by

$$\gamma(t) = \pi(2t, 3t), \qquad \pi : \mathbb{R}^2 \to M \ \text{ the natural quotient map.}$$

(1) Prove that the image of γ is an embedded submanifold of M.

(2) Let $H^1(M)$ denote the first de Rham cohomology group. Explain why the map

$$H^1(M) \ni [\alpha] \mapsto \int_\gamma \alpha \in \mathbb{R}$$

is well-defined, and describe its kernel.

Problem 4.- Let $O(n)$ be the orthogonal group of $n \times n$ real matrices, considered as a submanifold of the vector space $\mathcal{M}(n)$ of all $n \times n$ real matrices. You do not have to prove that it is a submanifold, but do not assume that its dimension is known. For each $g \in O(n)$ identify $T_g O(n)$ with a linear subspace of $\mathcal{M}(n)$.

(1) Explicitly compute $T_g O(n)$ for an arbitrary g. Prove your answer. What is its dimension?

(2) Let $F : O(n) \to O(n)$ be the map $F(k) = k^2$. Using part (1) compute $dF_g : T_g O(n) \to T_{g^2} O(n)$. Check that indeed your answer is in $T_{g^2} O(n)$.

Problem 5.- An *almost complex structure* J on M is an assignment of a linear map $J_x : T_x M \to T_x M$ to each $x \in M$ such that $\forall x \in M\ J_x^2 = -I$.

(1) What is the natural definition that such a J is smooth? This will be assumed for the rest of the problem.

(2) Given an almost complex structure J, its "Nijenhuis tensor" $\mathcal{N}$ is defined as follows: $\forall x \in M,\ \forall v, w \in T_x M$

$$\mathcal{N}_x(v, w) = ([JV, JW] - J[JV, W] - J[V, JW] - [V, W])_x \in T_x M$$

where V, W are smooth vector fields on M extending v, w, and the right-hand side is evaluated at x.

Show that the expression above does not depend on the choice of the extensions V, W, by showing that it is bilinear over $C^\infty(M)$.