ALGEBRAIC TOPOLOGY QR
AUGUST 2020

All maps below are assumed to be continuous.

(1) Let M be the Möbius band. Consider the pushout X of

$$\begin{array}{ccc} S^1 = \partial M & \longrightarrow & M \\ \downarrow & & \\ S^1 & & \end{array}$$

where the horizontal map is the inclusion of the boundary, and the vertical map is a degree 2 covering space. Describe each $H_i(X)$ as an abelian group.

(2) Consider the following property of spaces X equipped with a base point $x \in X$:

($*$) For any covering space $Y \to X$, if Y is connected, then so is the preimage $f^{-1}(X - \{x\})$.

For each of the following spaces X, determine if they satisfy ($*$) with respect to any base point. (If yes, then give a proof; if not, then give an example.)

(a) $X = S^1$.

(b) $X = \Sigma_2$ (the compact closed oriented surface of genus 2).

(3) Let $f : S^4 \to S^4$ be a map with the property that $f(x) = f(y)$ if y is the antipode of x. Show that $H_4(f) = 0$.

(4) Show that a finite group G of order 7 cannot act freely on $\mathbf{CP}^5$.

(5) For each of the following cases, determine if there exists a covering space $f : X \to Y$. (If yes, then construct it; if not, then give a proof.)

(a) X is homotopy equivalent to $S^1 \times S^1$ and Y homotopy equivalent to $S^1 \vee S^1$.

(b) X is homotopy equivalent to $S^1 \vee S^1$ and Y homotopy equivalent to $S^1 \times S^1$.

(1) As X is connected, we have $H_0(X) = \mathbf{Z}$. The long exact sequence for homology of the pushout gives a short exact sequence

$$0 \to H_1(S^1) \to H_1(M) \times H_1(S^1) \to H_1(X) \to 0$$

and shows $H_i(X) = 0$ for $i > 1$. To compute $H_1(X)$, we can identify the above sequence as

$$0 \to \mathbf{Z} \xrightarrow{1 \mapsto (a,b)} \mathbf{Z} \times \mathbf{Z} \to H_1(X) \to 0.$$

To compute (a, b), note that $a = 2$ since the natural homotopy equivalence $M \to S^1$ induces a degree 2 covering map $S^1 = \partial M \to S^1$. Moreover, $b = 2$ by assumption. So we learn that $H_1(X) = \mathbf{Z}^2/\langle(2, 2)\rangle \simeq \mathbf{Z} \times \mathbf{Z}/2$ with $(1, 0)$ and $(0, 1)$ on the right corresponding to $(1, -1)$ and $(1, 1)$ on the left.

(2) (a) False: the degree 2 connected cover $S^1 \to S^1$ becomes disconnected over $S^1 - \{x\} \subset S^1$ since $S^1 - \{x\} \simeq \mathbf{R}$ is simply connected.

(b) True: recall by covering space theory that a covering space $f : Y \to X$ is connected exactly when the $\pi_1(X)$-action on the fibre $f^{-1}(x)$ is transitive. It follows that if $\pi_1(X - \{x\}) \to \pi_1(X)$ is surjective, then the property $(*)$ is always true. This is the case for $X = \Sigma_2$: indeed, we know that $\Sigma_2 - \{x\} \subset \Sigma_2$ induces the surjection $\langle a, b, c, d\rangle \to \langle a, b, c, d \mid [a, b][c, d]\rangle$ by the standard calculation of these fundamental groups.

(3) The map f factors over the quotient $S^4 \to S^4/(x \sim a(x))$, where $a(x)$ is the antipode of x. But $S^4/(x \sim a(x))$ is another name for $\mathbf{RP}^4$, and we know that $H_4(\mathbf{RP}^4) = 0$, so $H_4(f)$ factors over the 0 group and is thus the 0 map.

(4) Assume towards contradiction that such an action exists. Let X be the quotient $\mathbf{CP}^5/G$, so X is a compact manifold and the map $\mathbf{CP}^5 \to X$ is a degree 7 covering space. But then $\chi(X) \cdot 7 = \chi(\mathbf{CP}^5)$. On the other hand, $\chi(\mathbf{CP}^5) = 6$ since $\mathbf{CP}^5$ has a CW decomposition with exactly 1 even dimensional cell up to dimension 6. So $\chi(X) = 6/7$, which is impossible as $\chi(X)$ must be an integer.

(5) (a) No: if such a covering space existed, then $\mathbf{Z}^2 = \pi_1(S^1 \times S^1)$ would be a subgroup of $F_2 := \mathbf{Z} * \mathbf{Z} = \pi_1(S^1 \vee S^1)$. But subgroups of free groups are free, so we would conclude that $\mathbf{Z}^2$ is free, which it is clearly not: it would have to free of rank 2 (as its abelianization is so), so the abelianness would imply that the free group of rank 2 is abelian, whence any pair of elements of any group commute with each other (by the universal property of free groups of rank 2), which is impossible by the existence of non-abelian groups.

(b) No: if such a covering space existed, then F_2 would be a subgroup of the abelian group $\mathbf{Z}^2$, whence F_2 would be abelian, which it is not by the reasoning in (a).