THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Topology

January 5, 2019: Algebraic Topology

1. Let a *quasi-torus* mean a torus from which a small disk (standard disk in a coordinate neighborhood) has been removed. Now consider three quasitori T_1, T_2, T_3. On each of their boundary circles, choose two distinct points A_i, B_i, $i = 1, 2, 3$, thus breaking up the boundary into two closed arcs $A_iB_i^+$ and $A_iB_i^-$. Now let X be a quotient of $T_1 \amalg T_2 \amalg T_3$ by attaching $A_1B_1^+$ to $A_2B_2^-$, $A_2B_2^+$ to $A_3B_3^-$, $A_3B_3^+$ to $A_1B_1^-$. Each pair of arcs is attached by identifying the arcs homeomorphically, with A_i going to A_j and B_i going to B_j. Compute the homology of X. Is X a topological manifold? If so, classify it.

2. Let S^1 be the unit circle in $\mathbb{C}$. Let Y be the space obtained from $S^1 \times [0, 1] \times \{0, 1\}$ (with the product topology, where each factor has the standard topology) by identifying $(z, 0, 0) \sim (z^3, 0, 1)$ and $(z, 1, 1) \sim (z^2, 1, 0)$. Calculate $\pi_1 Y$.

3. Let S_1, S_2 be two disjoint copies of the n-sphere, $n > 1$ fixed. Choose two distinct points $A_i, B_i \in S_i$. Let Z be a space obtained from $S_1 \amalg S_2$ by identifying $A_1 \sim A_2$, $B_1 \sim B_2$. Compute, with proof, the lowest possible number of cells in a CW-decomposition of Z.

4. Prove or disprove the following statement: Every compact surface whose universal cover is contractible has a regular cover of degree n for every $n \in \mathbb{N}$.

5. Is every (continuous) map from the real projective plane to the 2-sphere (with the standard topology) homotopic to a constant map? Prove your answer.

1. a)What are the possible one dimensional smooth connected manifolds (possibly with boundary) up to diffeomorphism? (No justification needed.)

b)Give an example (with proof) of a homeomorphism $\mathbb{R} \to \mathbb{R}$ which is not a diffeomorphism.

c)Construct a smooth structure R' on $\mathbb{R}$ such that the identity function on $\mathbb{R}$ is not a diffeomorphism, i.e. $(\mathbb{R}, R) \xrightarrow{\psi} (\mathbb{R}, R')$, such that $\psi_{\mathbb{R}} = id$, but ψ is not smooth.

2. Given a function $f : \mathbb{R}^n \to \mathbb{R}^m$ which is smooth, show that

$$\text{graph}(f) = \{(x, f(x)) \in \mathbb{R}^{n+m} : x \in \mathbb{R}^n\}$$

is a smooth submanifold of $\mathbb{R}^{n+m}$.

3. Let B^n be the n-dimensional ball and S^{n-1} ($n-1$ dimensional sphere) be the boundary of B^n. Is it possible to extend the identity map of S^{n-1} to a smooth map $B^n \to S^{n-1}$? Give an explicit map if possible, or prove it's impossible.

4. Suppose that a finite group G acts smoothly on a compact manifold M and that the action is free (i.e. if $\forall x \in M, g \circ x = x$, then g is the identity in G).
a) Show that M/G is a manifold.

b) Show that $M \to M/G$ is a covering space.

5. The special orthogonal group $SO(3) \subset M_3(\mathbb{R})$ is the collection of orthogonal matrices with determinant 1. Complete the following 4 steps (or find your own approach) to give a rigors construction of smooth coordinate charts on $SO(3)$

a) Define $exp(A) := \sum_{n=0}^{\infty} \frac{A^n}{n!} = 1 + A + A^2/2! + \cdots$. Prove that $exp(A)$ always converges, so that it's a map from $M_3(\mathbb{R})$ to $M_3(\mathbb{R})$.

b) Show that exp is injective on some neighbourhood of the zero matrix in $M_3(\mathbb{R})$. (Hint: You may assume that exp is smooth and use inverse function theorem.)

c) Prove if B is skew-symmetric, i.e. $B^\top = -B$, then $exp(B) \in SO(3)$.

d) Show that exp restricted to the space of skew-symmetric matrices is a surjective map (smooth submersion) onto $SO(3)$. (Hint: Euler's rotation theorem says that every 3-dim rotation has an axis.) Hence this gives $SO(3)$ a smooth chart.