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SOLUTIONS

1. Let M be a smooth manifold and $C \subset O \subset M$, where C is a closed smooth submanifold and O is an open subset. Show that if $f : C \to \mathbb{R}$ is a smooth function, then there is a smooth function $\hat{f} : M \to \mathbb{R}$ such that $\hat{f}|_C = f$ and $\text{supp}(\hat{f}) \subset O$.

Solution: The open sets O and $M \setminus C$ cover M. So there exists a C^∞ partition of unity subordinate to this open covering, i.e. smooth functions ρ_1, ρ_2 on M so that

(1) $0 \leq \rho_i \leq 1$
(2) $\text{supp}\rho_1 \subset O$, $\text{supp}\rho_2 \subset M \setminus C$
(3) $\rho_1(x) + \rho_2(x) = 1$ for all $x \in M$

Let $\tilde{f} = \rho_1 \cdot f$. This is well-defined since ρ_1 is 0 outside of O.

Since $\rho_1 + \rho_2 \equiv 1$ and ρ_2 is 0 on C (by condition 2), we must have $\rho_1 \equiv 1$ on C. So $\tilde{f} \equiv f$ on C.

2. Let M be an orientable manifold and let $\Psi : M \to \mathbb{R}$ be a smooth map. Show that if 0 is a regular value of Ψ, then $\Psi^{-1}(0) \subset M$ is also a smooth orientable manifold.

Solution: Let $n = \dim M$. By the regular value theorem, $\Psi^{-1}(0)$ is a smooth submanifold of M of dimension $n - 1$.

To show orientability, we will use that a manifold is orientable iff it admits a nowhere vanishing volume form. First, assume without loss of generality that M is an embedded submanifold of $\mathbb{R}^N$ for some N (Whitney embedding theorem). Then we can consider the vector field $\text{grad}_p\Psi$. Since 0 is a regular value of Ψ, we know this vector field is nowhere 0 for $p \in \Psi^{-1}(0)$. Moreover, the regular value theorem implies

$$T_p\Psi^{-1}(0) = \{v \in T_pM | \text{grad}_p\Psi \cdot v = 0\}.$$

Now let ω be the volume form on M. For $p \in \Psi^{-1}(0)$ and $v_1, \dots, v_{n-1} \in T_p\Psi^{-1}(0)$ define

$$\eta_p(v_1, \dots, v_{n-1}) = \omega_p(v_1, \dots, v_{n-1}, \text{grad}_p\Psi).$$

We claim the form η is nowhere zero. Since $\text{grad}_p\Psi$ is orthogonal to $T_p\Psi^{-1}(0)$, we know if $v_1, \dots, v_{n-1}$ is a basis for $T_p\Psi^{-1}(0)$, then $v_1, \dots, v_{n-1}, \text{grad}_p\Psi$ is a basis for T_pM. Since ω_p is a volume form, this means the righthand side of the above equation is nonzero for any $p \in \Psi^{-1}(0)$.

3.

(1) Give an example (with proof) of a homeomorphism $\mathbb{R} \to \mathbb{R}$ which is not a diffeomorphism.

Solution: Let $f(x) = x^{1/3}$. This is a continuous function on $\mathbb{R}$. Moreover, f is a bijection since it has an inverse given by $f^{-1}(x) = x^3$. This inverse is also continuous. Therefore, f is a homeomorphism.

However, f is not a diffeomorphism since it fails to be smooth at $x = 0$. Indeed, when $x \neq 0$ we have $f'(x) = x^{-2/3}/3$. This has no continuous extension to $x = 0$ since $f'(x)$ goes to infinity as x goes to 0, which shows f cannot be C^1.

(2) Construct a smooth structure R' on $\mathbb{R}$ such that the identity function on $\mathbb{R}$ is not a diffeomorphism, i.e. $\psi : (\mathbb{R}, R) \to (\mathbb{R}, R')$ such that $\psi_{\mathbb{R}} = id$, but ψ is not smooth.

Solution: Let $\psi : (\mathbb{R}, R) \to (\mathbb{R}, R')$ be given by $x \mapsto x^3$. The coordinate representation of the identity map from the standard smooth structure on $(\mathbb{R}, R)$ to this new smooth structure $(\mathbb{R}, R')$ is given by $x \mapsto x^{1/3}$, which is not smooth.

4. Consider the form $\omega = (x^2 + 2x + z)dy \wedge dz$ on $\mathbb{R}^3$. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere and $i : S^2 \hookrightarrow \mathbb{R}^3$ be the inclusion map.

(1) Evaluate the integral $\int_{S^2} i^*\omega$.

Solution: Let B^3 denote the unit ball in $\mathbb{R}^3$. By Stokes' theorem we have

$$\begin{aligned}\int_{S^2} i^*\omega &= \int_{B^3} d\omega \\ &= \int_{B^3} (2x + 2) dx \wedge dy \wedge dz \\ &= 2\text{vol}(B^3),\end{aligned}$$

where the $2x$ term vanishes because $2x$ is an odd function and the domain of integration B^3 is symmetric about the origin.

(2) Construct a closed form θ on $\mathbb{R}^3$ so that $i^*\theta = i^*\omega$, or prove no such form exists.

Solution: Suppose for contradiction such a 2-form θ exists. Then $\theta = d\alpha$ for some 1-form α on $\mathbb{R}^3$. Note $i^*\omega = i^*\theta = di^*\alpha$. Stokes theorem implies

$$\int_{S^2} i^*\omega = \int_{S^2} di^*\alpha = 0,$$

which contradicts the computation in the previous question.

5. Denote $\mathcal{M}_{m\times n}(\mathbb{R})$ the space of $m \times n$ matrices with real-valued entries. Show that the subset $\mathcal{S}_k \subset \mathcal{M}_{m\times n}(\mathbb{R})$ of rank k matrices forms a dimension $k(m + n - k)$ smooth submanifold of $\mathcal{M}_{m\times n}(\mathbb{R})$. Here $1 \leq k < m \leq n$.

Solution: We begin by defining a map

$$F : \mathbb{R}^{kn} \times \mathbb{R}^{k(m-k)} \to M_{m\times n}$$

as follows. Let $v_1, \ldots, v_k \in \mathbb{R}^n$ and let $\lambda_{i,j} \in \mathbb{R}$ for $k+1 \leq i \leq m$ and $1 \leq j \leq k$. Now let $F(v_1, \ldots, v_k, \lambda_{i,j})$ be the $m \times n$ matrix whose first k rows are given by $v_1, \ldots, v_n$ and the ith row for $k+1 \leq i \leq n$ is given by the linear combination $\sum_j \lambda_{i,j} v_j$.

If we restrict F to the open subset U of $\mathbb{R}^{kn} \times \mathbb{R}^{k(m-k)}$ corresponding $v_1, \ldots v_k$ linearly independent and not all $\lambda_{i,j}$ are zero, we obtain all rank k matrices where the first k rows are linearly independent and the remaining rows are linear combinations of the first k.

Now let $I = (i_1, \ldots, i_k)$ with $1 \leq i_1 < ... < i_k \leq m$. Suppose A is a rank k matrix whose row space is spanned by rows $i_1, \ldots, i_k$. Then there exists an $m \times m$ permutation matrix P_I such that $P_I A$ has the first k rows linearly independent. Then $P_I^{-1} F(v_1, \ldots, v_n, \lambda_{ij})$ gives all matrices with rows $i_1, \ldots, i_k$ linearly independent. So the collection of maps $P_I^{-1} \circ F|_U$ for all multi-indices I of length k gives a collection of charts covering $\mathcal{S}_k$. The transition functions are smooth since they are of the form $P_J P_I^{-1}$.