THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Topology

September 2018
Algebraic Topology

1. Consider two disjoint squares $ABCD$, $EFGH$ in $\mathbb{R}^2$. Identify their sides as follows:

$$AD \text{ with } HG,$$

$$DC \text{ with } EH,$$

$$AB \text{ with } BC,$$

$$EF \text{ with } FG.$$

All identifications of sides are bijective linear, with the endpoints identified in the order given. Is the quotient space of the identification a compact surface (i.e. a compact topological 2-manifold)? If so, classify it.

2. For which $n \in \mathbb{N}$ does there exist a CW structure on $\mathbb{C}P^{2n}$ with no cell in dimension n? Prove your answer.

3. Let
$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$
Let X be the quotient of $S^1 \times [0,1]$ by the smallest equivalence relation $\sim$ satisfying
$$(x,0) \sim (e^{2\pi i/3}x, 0),$$
$$(y,1) \sim (e^{2\pi i/6}y, 1)$$
for $x, y \in S^1$. Calculate $\pi_1(X)$.

4. Let
$$X_1 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\},$$
$$X_2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1 \text{ and } z = 0\},$$
$$X_3 = \{(x,y,z) \in \mathbb{R}^3 \mid x = y = 0 \text{ and } 0 \leq z \leq 1\}.$$
Let $X = X_1 \cup X_2 \cup X_3$, with the induced topology from $\mathbb{R}^3$. Describe the universal covering space $\widetilde{X}$ of X.

5. Let

$$S = \{(z,t) \in \mathbb{C}^2 \mid |z|^2 + |t|^2 = 1\},$$
$$T = \{(z,t) \in S \mid t^3 \in [0,\infty) \subset \mathbb{R}\}.$$

Let X (resp. Y) be the quotient of S (resp. T) by the equivalence relation identifying

$$(z_1, t_1) \sim (z_2, t_2)$$

when $z_1 = z_2\lambda$, $t_1 = t_2\lambda$ for some $\lambda \in \mathbb{C}$ with $\lambda^3 = 1$.

(a) Calculate the homology of Y.

(b) Calculate the homology of X.

1. Let S_1, S_2 be two smooth embedded submanifolds of manifold M.
1) Write down the definition for S_1, S_2 to be transversal.
2) Show that if $S_1, S_2 \subset M$ are transversal, then $S_1 \cap S_2 \subset M$ is a smooth embedded submanifold of dimension $dimS_1 + dimS_2 - dimM$.

2. Let M be a smooth orientable manifold and let $\Psi : M \to \mathbb{R}$ be a smooth map. Show that if 0 is a regular value of Ψ, then $\Psi^{-1}(0) \subset M$ is also a smooth orientable manifold.

3. Prove the following statement if it's true, or disprove using a counterexample: Let M and N be two smooth manifolds, if the tangent bundles TM and TN are diffeomorphic, then M and N are diffeomorphic.

4. Consider the form $\omega = (x^2 + 2x + 3y + 4z)dy \wedge dz$ on $\mathbb{R}^3$. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere and $\iota : S^2 \to \mathbb{R}^3$ be the inclusion map.
1) Evaluate the integral $\int_{S^2} \omega$.
2) Construct a closed form θ on $\mathbb{R}^3$ s.t. $\iota^*\theta = \iota^*\omega$, or prove that such a form θ does not exist.

5. Prove the following statements about Lie groups:

1) $SL_2(\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) : detA = 1\}$ is diffeomorphic to $S^1 \times \mathbb{R}^2$.
2) $SL_2(\mathbb{C}) = \{A \in M_{2\times 2}(\mathbb{C}) : detA = 1\}$ is diffeomorphic to $S^3 \times \mathbb{R}^3$.