May 8, 2018

1. Let $M^m \subset \mathbb{R}^n$ be a smooth submanifold of dimension $m < n - 2$. Show that its complement $\mathbb{R}^n \setminus M$ is connected and simply connected.

2. Let $\alpha$ be a closed differential two-form on $S^4$. Prove that $\alpha \wedge \alpha$ vanishes at some point.

3. Show that the real cubic surface defined by $S = \{[x : y : z : w] \in \mathbb{R}P^3 : x^3 + y^3 + z^3 + w^3 = 0\}$ is an embedded submanifold of $\mathbb{R}P^3$, and compute its (real) dimension.

4. Consider the following subgroup $H$ of $GL_2(\mathbb{R})$:

$$H = \{h \in GL_2(\mathbb{R}), h = \begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix}, u > 0, v \in \mathbb{R}\}.$$ 

Show that the vector fields $u \frac{\partial}{\partial u}$ and $u \frac{\partial}{\partial v}$ form a basis of the Lie algebra $\mathfrak{h}$ of $H$.

5. Let $M$ be a smooth manifold and $C \subset O \subset M$, where $C$ is a closed subset and $O$ is an open subset. Let $f : C \to \mathbb{R}$ be a smooth function, which means $\forall p \in C, \exists$ an open set $p \in V_p \subset M$ and a smooth function $\hat{f}_p : C \to \mathbb{R}$ s.t. $\hat{f}_p|_{C \cap V_p} = f|_{C \cap V_p}$.

a. Show that there is a smooth function $\hat{f} : M \to \mathbb{R}$, such that $\hat{f}|_C = f$ and $\text{supp} (\hat{f}) \subset O$.

b. If the set $C$ is not assumed to be closed, then does the statement of part a) still hold? If yes, give the proof; and if not, give a counterexample.
1. Let $X$ be the space obtained by removing the open square in $\mathbb{R}^2$ with vertices $(11), (12), (21), (22)$ from the closed square with vertices $(00), (03), (30), (33)$. Now let $X$ be the space obtained by identifying the following pairs of line segments, direction indicated, via affine bijective maps:

   (00), (03) with (21), (22),
   (30)(33) with (11), (12),
   (00), (30) with (22), (12),
   (03), (33) with (21), (11).

   (a) Calculate $\pi_1(X)$.
   (b) Prove that $X$ is a compact surface, and classify it.

2. For which $n \in \mathbb{N}$ does there exist a CW structure on $\mathbb{C}P^{2n}$ with no cell in dimension $2n$? Prove your answer.

3. Let

   \[ S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}, \]
   \[ D^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}. \]

   Let $f : S^2 \to S^2$ be a (continuous) map of degree $k$. Let $X$ be the pushout of the diagram

   \[
   \begin{array}{ccc}
   S^2 & \xrightarrow{f} & S^2 \\
   \downarrow{c} & & \downarrow{c} \\
   D^3 & & \\
   \end{array}
   \]

   Calculate the homology of $X$.

4. Consider the 1-point compactification $X = \mathbb{R}^3 \cup \{\infty\}$ of $\mathbb{R}^3$. Now let $\mathbb{Z}/2$ act on $X$ where the generator sends $x \mapsto -x$ for $x \in \mathbb{R}^3$, and $\infty \mapsto \infty$. Let $Y$ be the orbit space of $X$ with the quotient topology. How many non-isomorphic connected covering spaces (in the unbased sense) does $Y$ have? Prove your answer.
5. Let $X$ be the pushout of the diagram

$$
\begin{array}{c}
S^1 \times S^1 \\
g \\
S^1
\end{array} \xrightarrow{f} S^1 \\
\downarrow \\
\downarrow g
$$

where $f$ is the projection to the first coordinate composed with a map of degree $k$, and $g$ is the projection to the second coordinate composed with a map of degree $\ell$. Compute $\pi_1(X)$. 