Topology Qualified Exam (Differential Part) at May, 2018

May 8, 2018

1. Let $M^m \subset \mathbb{R}^n$ be a smooth submanifold of dimension $m < n-2$. Show that its complement $\mathbb{R}^n \setminus M$ is connected and simply connected.

2. Let α be a closed differential two-form on S^4. Prove that $\alpha \wedge \alpha$ vanishes at some point.

3. Show that the real cubic surface defined by $S = \{[x : y : z : w] \in \mathbb{R}P^3 : x^3 + y^3 + z^3 + w^3 = 0\}$ is an embedded submanifold of $\mathbb{R}P^3$, and compute its (real) dimension.

4. Consider the following subgroup H of $GL_2(\mathbb{R})$:

$$H = \{h \in GL_2(\mathbb{R}), h = \begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix}, u > 0, v \in \mathbb{R}\}.$$

Show that the vector fields $u\frac{\partial}{\partial u}$ and $u\frac{\partial}{\partial v}$ form a basis of the Lie algebra $\mathfrak{h}$ of H.

5. Let M be a smooth manifold and $C \subset O \subset M$, where C is a closed subset and O is an open subset. Let $f : C \to \mathbb{R}$ be a smooth function, which means $\forall p \in C$, $\exists$ an open set $p \in V_p \subset M$ and a smooth function $\hat{f}_p : C \to \mathbb{R}$ s.t. $\hat{f}_p|_{C \cap V_p} = f|_{C \cap V_p}$.

a. Show that there is a smooth function $\hat{f} : M \to \mathbb{R}$, such that $\hat{f}|_C = f$ and $supp(\hat{f}) \subset O$.

b. If the set C is not assumed to be closed, then does the statement of part a) still hold? If yes, give the proof; and if not, give a counterexample.

THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Topology

May 2018
Algebraic Topology

1. Let X be the space obtained by removing the open square in $\mathbb{R}^2$ with vertices $(11), (12), (21), (22)$ from the closed square with vertices $(00), (03), (30), (33)$. Now let X be the space obtained by identifying the following pairs of line segments, direction indicated, via affine bijective maps:

   $(00), (03)$ with $(21), (22)$,

   $(30)(33)$ with $(11), (12)$,

   $(00), (30)$ with $(22), (12)$,

   $(03), (33)$ with $(21), (11)$.

   (a) Calculate $\pi_1(X)$.

   (b) Prove that X is a compact surface, and classify it.

2. For which $n \in \mathbb{N}$ does there exist a CW structure on $\mathbb{C}P^{2n}$ with no cell in dimension $2n$? Prove your answer.

3. Let
$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\},$$
$$D^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\},$$
   Let $f : S^2 \to S^2$ be a (continuous) map of degree k. Let X be the pushout of the diagram
$$\begin{array}{ccc} S^2 & \xrightarrow{f} & S^2 \\ {\scriptstyle\subset}\big\downarrow & & \\ D^3. & & \end{array}$$
   Calculate the homology of X.

4. Consider the 1-point compactification $X = \mathbb{R}^3 \cup \{\infty\}$ of $\mathbb{R}^3$. Now let $\mathbb{Z}/2$ act on X where the generator sends $x \mapsto -x$ for $x \in \mathbb{R}^3$, and $\infty \mapsto \infty$. Let Y be the orbit space of X with the quotient topology. How many non-isomorphic connected covering spaces (in the unbased sense) does Y have? Prove your answer.

5. Let X be the pushout of the diagram

$$\begin{array}{ccc} S^1 \times S^1 & \xrightarrow{f} & S^1 \\ {\scriptstyle g}\downarrow & & \\ S^1 & & \end{array}$$

where f is the projection to the first coordinate composed with a map of degree k, and g is the projection to the second coordinate composed with a map of degree ℓ. Compute $\pi_1(X)$.