1. Show that
\[ \omega = \frac{-y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy \]
defines a nonzero deRham cohomology class of \( \mathbb{R}^2 \setminus \{0,0\} \).

Solution: We need to show \( \omega \) is closed but not exact.

Computing the exterior derivative gives
\[ d\omega = \left( \frac{\partial}{\partial x} \frac{x}{x^2 + y^2} - \frac{\partial}{\partial y} \frac{-y}{x^2 + y^2} \right) \, dx \wedge dy \]
\[ = \left( \frac{x^2 + y^2}{(x^2 + y^2)^2} - 2x^2 + \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} \right) \, dx \wedge dy \]
\[ = 0, \]

so \( \omega \) is closed.

To show \( \omega \) is not exact, we integrate \( \omega \) around the unit circle \( C \), which we will parametrize by \( f(t) = (\cos(t), \sin(t)) \) from \( t = 0 \) to \( 2\pi \). It suffices to show the integral \( \int_C \omega \neq 0 \). Indeed, if \( \omega \) were exact, then \( \omega = dg \) for some smooth function \( g \) and then Stokes’ theorem would give \( \int_C dg = \int_{\partial C} g = 0 \) since \( C \) has no boundary.

We have
\[ f^*dx = -\sin(t) dt \]
\[ f^*dy = \cos(t) dt. \]

Writing \( x, y, dx, dy \) in terms of \( t \), we get
\[ \int_C \omega = \int_0^{2\pi} \sin^2(t) + \cos^2(t) \, dt = 2\pi \neq 0. \]

Therefore, \( \omega \) is not exact.

2. Any non-constant smooth function of a compact connected manifold has at least two critical points.

Solution: Let \( M \) be a compact connected manifold and let \( f : M \to \mathbb{R} \) be a smooth function. Since \( M \) is compact and \( f \) is continuous, the function \( f \) must achieve both a maximum and minimum value on \( M \). Suppose \( p \) is such that \( f(p) \) is the maximum value of \( f \). We claim \( f \) has a critical point at \( p \). To see this, let \( \gamma : (-\varepsilon, \varepsilon) \to M \) be a curve with \( \gamma(0) = p \) (this is assuming \( M \) has no boundary, which is necessary or else \( f(x) = x \) on \([0,1]\) is a counter-example to the statement of the problem). Suppose for contradiction \( \gamma'(t) \neq 0 \). If \( \gamma'(t) > 0 \) then there is small enough \( t > 0 \) so that \( f(\gamma(t)) > f(p) \) and if \( \gamma'(0) < 0 \), there is \( t > 0 \) so that \( f(\gamma(-t)) > f(p) \), contradicting the maximality of \( f(p) \). A similar argument
shows that if \( q \in M \) is such that \( f(q) \) is the minimum value at \( f \), then \( f \) must have a critical point at \( q \).

3. For each \( n \geq 1 \), there is a diffeomorphism \((TS^n) \times \mathbb{R} \cong S^n \times \mathbb{R}^{n+1}\).

Solution: View \( S^n \) as \( \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_n^2 = 1\} \). Let \( N_p \) be the normal vector field to \( S^n \), i.e. associate to each \( p = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^n \) the vector \((x_1, \ldots, x_{n+1})\). Define

\[
F : (TS^n) \times \mathbb{R} \to S^n \times \mathbb{R}^{n+1}
\]

\[
(p, v, \alpha) \mapsto (p, v + \alpha N_p).
\]

This map is clearly smooth. It is bijective because any \( w \in \mathbb{R}^{n+1} \) can be decomposed uniquely as \( w = w_1 + w_2 \) where \( w_1 \) is in the direction of \( N_p \) and \( w_2 \) is in the direction normal to \( N_p \), i.e. in the direction of \( T_p S^n \).

4. Assuming that every \( n \)-dimensional compact manifold \( M^n \) can be embedded into some \( \mathbb{R}^N \), prove that we can choose \( N = 2n + 1 \). (Hint: Given a nonzero vector \( v \neq 0 \) in \( \mathbb{R}^N \), once can define a parallel projection \( \phi_v \) from \( \mathbb{R}^N \) to the orthogonal complement of \( v \). If \( N > 2n + 1 \), we can choose some \( v \) so that \( \phi_v \mid M^n \) is an embedding.)

Solution: We need to choose \( v \) so that \( \phi_v \) is an injective immersion.

Injective: In order for \( \phi_v \) to be injective, we need the line \( tv \) to pass through at most one point on \( M \). Define

\[
F : M \times M \setminus \Delta \to \mathbb{R}P^{N-1}
\]

\[
(p, q) \mapsto [p - q].
\]

If \( F(p, q) = [v] \), then the line \( tv \) will pass through both \( p \) and \( q \), which means \( \phi_v \) is not injective. If \([v]\) is not in the image of \( F \), then the fiber of \( \phi_v \) contains at most one point, i.e. \( \phi \) is injective. We claim the image of \( F \) has measure 0 so long as \( 2n = \dim(M \times M \setminus \Delta) > \dim(\mathbb{R}P^{N-1}) = N - 1 \). Indeed, if the dimensions satisfy this inequality, \( F \) cannot have any regular values, so every value of \( F \) must be a critical value. The set of critical values has measure 0 by Sard’s theorem.

Immersion: In order for \( \phi_v \) to be an immersion, we must have \( d\phi_v \) is injective on tangent spaces. Since \( \phi_v \) is linear, its derivative is itself. For the projection \( \phi_v \) to be injective on tangent spaces, we must have that \( v \) is not parallel to any vector tangent to \( M \). Define

\[
G : TM \setminus \{(p, 0) \mid p \in M\} \to \mathbb{R}P^{N-1}
\]

\[
(p, v) \mapsto [v].
\]

If \( v \) is not in the image of \( G \), then \( v \) is not parallel to any vector tangent to \( M \). Again, by Sard’s theorem, the image of \( G \) has measure 0 so long as \( 2n = \dim(TM) > \dim(\mathbb{R}P^{N-1}) = N - 1 \).

In conclusion, if \( 2n > N - 1 \), we can choose \( v \) so that \([v] \in \mathbb{R}P^{N-1}\) is not in the image of \( F \) or \( G \), guaranteeing the projection \( \phi_v \) is an injective immersion, and hence an embedding.

5.
(1) Show that the space of orthogonal matrices
\[ O(n) = \{ A \in M_{n \times n}(\mathbb{R}) \mid AA^T = I \} \]
is a smooth submanifold of \( M_{n \times n}(\mathbb{R}) \).

Solution: Let \( S(n) \) denote the space of \( n \times n \) real symmetric matrices. Consider the function
\[
F : M_{n \times n}(\mathbb{R}) \to S(n)
\]
\[
A \mapsto AA^T.
\]
By the regular value theorem, it suffices to show \( I \) is a regular value of \( F \). To this end, we compute the derivative
\[
DF_A : T_A M_{n \times n}(\mathbb{R}) \to T_{Id} S(n)
\]
for any \( A \in O(n) \) and show it is surjective. Note we can identify \( T_A M_{n \times n}(\mathbb{R}) \cong M_{n \times n}(\mathbb{R}) \) and \( T_F(A)S(n) \cong S(n) \). Let \( B \in T_A M_{n \times n}(\mathbb{R}) \cong M_{n \times n}(\mathbb{R}) \). Then
\[
DF_A(B) = \frac{d}{dt} \bigg|_{t=0} F(A + tB)
\]
\[
= \frac{d}{dt} \bigg|_{t=0} (A + tB)(A + tB)^T
\]
\[
= \frac{d}{dt} \bigg|_{t=0} AA^T + t(AB^T + BA^T) + t^2(BB^T)
\]
\[
= AB^T + BA^T.
\]
To see \( DF_A \) is surjective onto symmetric matrices, note that we can write any symmetric matrix \( X \) as \( P + P^T \), where \( P \) is upper triangular. (Indeed, take the super diagonal entries of \( P \) to be the same as \( A \), and take the diagonal entries of \( P \) to be one half times the diagonal entries of \( A \)). Then given \( P \) and \( A \), we can solve the equation \( P = AB^T \) for \( B \), since we are assuming \( A \in O(n) \) and so \( A \) is invertible because \( A^{-1} = A^T \). This shows \( DF_A \) is surjective as desired.

(2) Verify that the tangent space at the identity matrix
\[ o(n) = \{ A \in M_{n \times n}(\mathbb{R}) \mid A + A^T = 0 \}. \]

Solution: First we claim these two vector spaces have the same dimension. The dimension of the right hand side is \( \frac{n^2 - n}{2} \), since we are free to specify the upper-triangular part of the matrix \( A \). To compute the dimension of the left hand side, we can use the regular value theorem. Since \( O(n) = F^{-1}(I) \) with \( F \) defined as above, we know \( O(n) \subset M_{n \times n}(\mathbb{R}) \) is a smooth submanifold with codimension equal to \( \dim S(n) = \frac{n^2 - n}{2} + n \). Hence \( \dim O(n) = \dim o(n) = n^2 - \dim S(n) = \frac{n^2 - n}{2}. \)

Next, suppose \( \gamma(t) \) is a curve in \( O(n) \) with \( \gamma(0) = I \), so \( \gamma'(0) \in o(n) \). Then
\[
F(\gamma(t)) = I.
\]
Differentiating both sides with respect to \( t \) and using the chain rule gives
\[
DF_{Id}(\gamma'(0)) = 0.
\]
This means
\[ o(n) \subset \ker DF_{Id} = \{ A \in M_{n \times n}(\mathbb{R}) \mid A + A^T = 0 \}. \]

Since these two vector spaces have the same dimension, the only way for one of them to be a subset of the other is if they are the same.

(3) Show that the tangent bundle \( TO(n) \) can be trivialized, i.e.
\[ TO(n) \cong O(n) \times o(n). \]

**Solution:** Let \( (g, v) \in TO(n), \) i.e. \( g \in O(n) \) and \( v \in T_g O(n) \). Let \( L_{g^{-1}} \) denote left multiplication by \( g^{-1} \). This map is a diffeomorphism, so its derivative at \( g \)
\[ D_g L_{g^{-1}} : T_g O(n) \to T_{Id} O(n) = o(n) \]
is a linear isomorphism. Let
\[ \Phi : TO(n) \to O(n) \times o(n) \]
\[ (g, v) \mapsto (g, D_g L_{g^{-1}}(v)). \]
Then \( \Phi \) is a diffeomorphism.