THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Topology

May 5, 2017: Morning Session, 9:00 to 12:00 noon.

1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by
$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 + 2xy + y^2 \\ 1 + 2xy + x^2 \end{pmatrix}.$$
(a) Prove that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a regular value of f.
(b) Prove that there exist $K > 0, \epsilon > 0$ such that for all $z \in \mathbb{R}^2$, $||z|| \geq K$ implies $||f(x)|| \geq \epsilon$.
(c) Thus, we can define a map
$$g : S^2 \cong \mathbb{R}^2/\{z \in \mathbb{R}^2 \mid ||z|| \geq K\} \to \mathbb{R}^2/\{z \in \mathbb{R}^2 \mid ||z|| \geq \epsilon\} \cong S^2$$
by $g : z \mapsto f(z)$. Calculate the degree of g, assuming we choose the homeomorphisms with S^2 in such a way that the map $z \mapsto z$ has degree 1.

2. Let $X = \mathbb{R}^3 \smallsetminus \{(x, y, z) \in \mathbb{R}^3 \mid y = 0 \text{ and } x^2 + z^2 = 1\}$. Prove that the differential form
$$\frac{2(1 - x^2 + y^2 - z^2)dy + 4xydx + 4yzdz}{(1 - x^2 - y^2 - z^2)^2 + 4y^2}$$
represents a nonzero element of $H^1_{DR}(X)$.

3. (a) Prove that the set of all $n \times n$ diagonalizable real matrices with given eigenvalues and multiplicities is a smooth submanifold of $\mathbb{R}^{n^2}$.
(b) Prove that the set of all 2×2 real matrices with double eigenvalue 0 is not a smooth submanifold of $\mathbb{R}^4$.

4. Is the set of all points in $\mathbb{R}^3$ satisfying the equations
$$y^3 + x^2 + z^4 = 3,$$
$$xyz = 1$$
a smooth submanifold of $\mathbb{R}^3$? Explain.

5. Prove or disprove the following statement: If $f : M \to N$ is a smooth diffeomorphism of smooth manifolds and u, v are vector fields on M, then $[Df(u), Df(v)] = Df([u, v])$.

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Qualifying Review examination in Topology

May 5, 2017: Afternoon Session, 2:00 to 5:00.

1. Let X be a connected CW-complex whose fundamental group is Σ_3, the group of all permutations on 3 elements.

   (a) How many isomorphism classes of objects are there in the category $Cov_0(X)$ of connected covering spaces of X and continuous maps commuting with the covering map?

   (b) How many isomorphism classes of objects of $Cov_0(X)$ have degree 2?

   (c) How many isomorphism classes of objects of $Cov_0(X)$ are regular coverings?

2. Let X be a connected CW-complex such that $H_i(X) = 0$ for all $i > 0$. Let S^k denote the k-sphere. Prove that for all $k \in \mathbb{N}$, $H_n(X \times S^k)$ is $\mathbb{Z}$ for $n = 0$ and $n = k$, and 0 for all other values of n.

3. Let F be the free group on a, b. Let $G = \{1, x, x^2\}$ be the cyclic group on three generators written multiplicatively. Let $h : F \to G$ be a homomorphism which sends $a \mapsto x$, $b \mapsto x^2$. Find free generators of $Ker(h)$.

4. For which connected compact surfaces X without boundary does there exist a continuous map $f : X \to X$ with no fixed point? [Hint: Verify by inspection that if $\mathbb{Z}$ is a direct summand of $H_1(X)$, then S^1 is a retract of X.]

5. Let S^1 be the set of complex numbers of absolute value 1 with the induced topology. K be the quotient space formed from $S^1 \times [0, 1]$ by identifying every point $(z, 0)$ with the point $(z^{-2}, 1)$. Compute the homology of K.