THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Topology

January 8, 2017: Morning Session, 9:00 to 12:00 noon.

1. Let M be a connected smooth manifold of dimension n, and let $\Lambda := \Lambda^n T^* M$ denote the real line bundle of forms of degree n. Let $\pi : \Lambda \to M$ be the projection onto M, and let $\Lambda_* = \Lambda \setminus 0_\Lambda$, where 0_Λ here denotes the zero section of Λ. The positive real numbers $\mathbb{R}_+$ act on Λ_* by scalar multiplication. Let $\mathcal{T}_M$ denote the quotient space $\Lambda_*/\mathbb{R}_+$ with the quotient topology.

   a. Show that $\mathcal{T}_M$ is a smooth manifold of dimension n.

   b. Show that π induces a map $\pi_{\mathcal{T}} : \mathcal{T}_M \to M$, and that $\pi_{\mathcal{T}}$ is a smooth covering map.

   c. Show that $\mathcal{T}_M$ is connected if and only if M is non-orientable.

2. Let M be a compact, connected smooth n-dimensional manifold, and let f be a smooth, real-valued function on M. The exterior derivative df defines a section of the cotangent bundle T^*M over M. Let $\Gamma_{df} \subset T^*M$ denote the graph of this section. It is a submanifold of dimension n. Say that the function f is *generic* if Γ_{df} intersects the zero-section $0_M \subset T^*M$ transversally.

   a. If f is generic and $x_0 \in M$ is in $\Gamma_{df} \cap 0_M$, where we identify M with the zero-section 0_M, then show that in any local coordinate system $x_1, \ldots, x_n$ centered at x_0, one has

   $$\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) \neq 0.$$

   b. If f is generic on M which is compact, then show that f has a finite number of critical points.

3. Let V be an 4-dimensional real vector space. Let $\omega \neq 0 \in \Lambda^4 V \cong \mathbb{R}$. Let $\alpha \in \Lambda^2 V$. For $\beta, \gamma \in V$, define a real-valued bilinear form λ_α on V by

   $$\Lambda^4 V \ni \alpha \wedge \beta \wedge \gamma = \lambda_\alpha(\beta, \gamma) \cdot \omega$$

   a. Show that λ_α has rank 0, 2 or 4.

   b. Show that the rank of λ_α is 2 if and only if there are two vectors $\phi_1, \phi_2 \in V$ such that $\alpha = \phi_1 \wedge \phi_2$.

c. For $u \in V^*$, the dual of V, define the *contraction* with u to be the linear map

$$i_u : \Lambda^i V \to \Lambda^{i-1} V$$

given by

$$i_u(v_1 \wedge \ldots \wedge v_i) = \sum (-1)^{j-1}\, u(v_j)\; v_1 \wedge \ldots \wedge v_{j-1} \wedge v_{j+1} \wedge \ldots \wedge v_i$$

for such elements, and extended to all of $\Lambda^i V$ by linearity. Show that for any non-zero $\beta \in \Lambda^2 V$, there is a $u \in V^*$ such that $i_u(\beta) \neq 0$ in $\Lambda^1 V = V$.

d. Show that the rank of λ_α is 2 if and only if $\alpha \neq 0$, and $\alpha \wedge \alpha = 0$.

4. Show that the real projective space $\mathbb{RP}^3$ is a Lie group, while $\mathbb{RP}^4$ is not a Lie group.

5. Let T^2 be the 2-dimensional torus $\cong \mathbb{R}^2/\mathbb{Z}^2$, where $\mathbb{R} = \{(x, y) \,|\, x, y \in \mathbb{R}\}$, and $\mathbb{Z}^2 = \{(m, n) \,|\, m, n \in \mathbb{Z}\}$. Let $\phi(t)$ be a smooth function on $\mathbb{R}$ which is increasing, and such that $\phi \equiv 0$, for $t \leq -\frac{1}{4}$, and $\phi(t) \equiv 1$, for $t \geq \frac{1}{4}$. Define one-forms $\alpha = \sum_{m \in \mathbb{Z}} d\phi(x + m)$, $\beta = \sum_{n \in \mathbb{Z}} d\phi(y + n)$ on $\mathbb{R}^2$.

a. Show that α and β are well-defined on $\mathbb{R}^2$ and are invariant under translation by $\mathbb{Z}^2$. Hence they define 1-forms, still denoted α, β, on the quotient manifold T^2.

b. Show that the forms α and β on T^2 are closed.

c. Let $[\alpha]$ and $[\beta]$ denote the corresponding de Rham cohomology classes in the first de Rham cohomology group $H^1_{dR}(T^2)$. Given that $H^1_{dR}(T^2) \cong \mathbb{R}^2$, show that $H^1_{dR}(T^2) \cong \mathbb{R} \cdot [\alpha] + \mathbb{R} \cdot [\beta]$.

THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Topology

January 8, 2017: Afternoon Session, 2:00 to 5:00.

1. Let $A_k = e^{2k\pi i/2n}$. Let C_n be the convex hull of $\{A_k \mid k = 0, 1, \ldots, 2n-1\}$ with the topology induced from $\mathbb{C}$. Let $\sim$ be the smallest equivalence relation on C_n such that

$$tA_k + (1-t)A_{k+1} \sim (1-t)A_{k+n} + tA_{k+n+1}, \text{ for all } k \in \mathbb{Z}/2n,\ 0 \leq t \leq 1.$$

Let $X_n = C_n/\sim$ with the quotient topology.

(a) Calculate $\pi_1(X_n)$.

(b) Classify the surface X_n.

2. Prove that the usual inclusions $\mathbb{C}P^0 \subset \mathbb{C}P^1 \subset \ldots \subset \mathbb{C}P^n$ define a CW filtration on $\mathbb{C}P^n$.

3. Let F be the free group on a, b. Let G be a symmetric group (=group of all permutations) on three elements, and let $x, y \in G$ be elements of order 2 and 3, respectively. Let $h : F \to G$ be a homomorphism which sends $a \mapsto x$, $b \mapsto y$. Find free generators of $Ker(h)$.

4. Compute the homology of the special orthogonal group $SO(3)$.

5. Let

$$S^2 = \{\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\},$$
$$D^3 = \{\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\},$$
$$Q = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \text{ or } z = 0 \text{ and } x^2 + y^2 \leq 1\}.$$

Let $f : S^2 \to S^2$ be a (continuous) map of degree k. Let X be the pushout of the diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{\subset \circ f} & Q \\ \subset \big\downarrow & & \\ D^3. & & \end{array}$$

Calculate the homology of X.