1. Let $M$ be a connected smooth manifold of dimension $n$, and let $\Lambda := \Lambda^n T^* M$ denote the real line bundle of forms of degree $n$. Let $\pi : \Lambda \rightarrow M$ be the projection onto $M$, and let $\Lambda_* = \Lambda \setminus 0_\Lambda$, where $0_\Lambda$ here denotes the zero section of $\Lambda$. The positive real numbers $\mathbb{R}_+$ act on $\Lambda_*$ by scalar multiplication. Let $T_M$ denote the quotient space $\Lambda_*/\mathbb{R}_+$ with the quotient topology.

a. Show that $T_M$ is a smooth manifold of dimension $n$.

b. Show that $\pi$ induces a map $\pi_T : T_M \rightarrow M$, and that $\pi_T$ is a smooth covering map.

c. Show that $T_M$ is connected if and only if $M$ is non-orientable.

2. Let $M$ be a compact, connected smooth $n$-dimensional manifold, and let $f$ be a smooth, real-valued function on $M$. The exterior derivative $df$ defines a section of the cotangent bundle $T^* M$ over $M$. Let $\Gamma_{df} \subset T^* M$ denote the graph of this section. It is a submanifold of dimension $n$. Say that the function $f$ is generic if $\Gamma_{df}$ intersects the zero-section $0_M \subset T^* M$ transversally.

a. If $f$ is generic and $x_0 \in M$ is in $\Gamma_{df} \cap 0_M$, where we identify $M$ with the zero-section $0_M$, then show that in any local coordinate system $x_1, \ldots, x_n$ centered at $x_0$, one has

$$\det \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right) \neq 0.$$ 

b. If $f$ is generic on $M$ which is compact, then show that $f$ has a finite number of critical points.

3. Let $V$ be an 4-dimensional real vector space. Let $\omega \neq 0 \in \Lambda^4 V \cong \mathbb{R}$. Let $\alpha \in \Lambda^2 V$. For $\beta, \gamma \in V$, define a real-valued bilinear form $\lambda_\alpha$ on $V$ by

$$\Lambda^4 V \ni \alpha \wedge \beta \wedge \gamma = \lambda_\alpha(\beta, \gamma) \cdot \omega$$

a. Show that $\lambda_\alpha$ has rank 0, 2 or 4.

b. Show that the rank of $\lambda_\alpha$ is 2 if and only if there are two vectors $\phi_1, \phi_2 \in V$ such that $\alpha = \phi_1 \wedge \phi_2$. 

c. For \( u \in V^* \), the dual of \( V \), define the *contraction* with \( u \) to be the linear map

\[ i_u : \Lambda^i V \to \Lambda^{i-1} V \]

given by

\[ i_u(v_1 \wedge \ldots \wedge v_i) = \sum (-1)^{j-1} u(v_j) \, v_1 \wedge \ldots \wedge v_{j-1} \wedge v_{j+1} \wedge \ldots \wedge v_i \]

for such elements, and extended to all of \( \Lambda^i V \) by linearity. Show that for any non-zero \( \beta \in \Lambda^2 V \), there is a \( u \in V^* \) such that \( i_u(\beta) \neq 0 \) in \( \Lambda^1 V = V \).

d. Show that the rank of \( \lambda_\alpha \) is 2 if and only if \( \alpha \neq 0 \), and \( \alpha \wedge \alpha = 0 \).

4. Show that the real projective space \( \mathbb{RP}^3 \) is a Lie group, while \( \mathbb{RP}^4 \) is not a Lie group.

5. Let \( T^2 \) be the 2-dimensional torus \( \cong \mathbb{R}^2 / \mathbb{Z}^2 \), where \( \mathbb{R} = \{(x, y) | x, y \in \mathbb{R}\} \), and \( \mathbb{Z}^2 = \{(m, n) | m, n \in \mathbb{Z}\} \). Let \( \phi(t) \) be a smooth function on \( \mathbb{R} \) which is increasing, and such that \( \phi \equiv 0 \), for \( t \leq -\frac{1}{4} \), and \( \phi(t) \equiv 1 \), for \( t \geq \frac{1}{4} \). Define one-forms \( \alpha = \sum_{m \in \mathbb{Z}} d\phi(x + m), \beta = \sum_{n \in \mathbb{Z}} d\phi(y + n) \) on \( \mathbb{R}^2 \).

a. Show that \( \alpha \) and \( \beta \) are well-defined on \( \mathbb{R}^2 \) and are invariant under translation by \( \mathbb{Z}^2 \). Hence they define 1-forms, still denoted \( \alpha, \beta \), on the quotient manifold \( T^2 \).

b. Show that the forms \( \alpha \) and \( \beta \) on \( T^2 \) are closed.

c. Let \([\alpha]\) and \([\beta]\) denote the corresponding de Rham cohomology classes in the first de Rham cohomology group \( H^1_{dR}(T^2) \). Given that \( H^1_{dR}(T^2) \cong \mathbb{R}^2 \), show that \( H^1_{dR}(T^2) \cong \mathbb{R} \cdot [\alpha] + \mathbb{R} \cdot [\beta] \).
1. Let \( A_k = e^{2k\pi i / 2n} \). Let \( C_n \) be the convex hull of \( \{ A_k \mid k = 0, 1, \ldots, 2n - 1 \} \) with the topology induced from \( \mathbb{C} \). Let \( \sim \) be the smallest equivalence relation on \( C_n \) such that 
\[
tA_k + (1 - t)A_{k+1} \sim (1 - t)A_{k+n} + tA_{k+n+1}, \quad \text{for all } k \in \mathbb{Z}/2n, \ 0 \leq t \leq 1.
\]
Let \( X_n = C_n / \sim \) with the quotient topology.

(a) Calculate \( \pi_1(X_n) \).

(b) Classify the surface \( X_n \).

2. Prove that the usual inclusions \( \mathbb{C}P^0 \subset \mathbb{C}P^1 \subset \cdots \subset \mathbb{C}P^n \) define a CW filtration on \( \mathbb{C}P^n \).

3. Let \( F \) be the free group on \( a, b \). Let \( G \) be a symmetric group (=group of all permutations) on three elements, and let \( x, y \in G \) be elements of order 2 and 3, respectively. Let \( h : F \rightarrow G \) be a homomorphism which sends \( a \mapsto x, \ b \mapsto y \). Find free generators of \( \ker(h) \).

4. Compute the homology of the special orthogonal group \( SO(3) \).

5. Let 
\[
S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \}, \\
D^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1 \}, \\
Q = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \text{ or } z = 0 \text{ and } x^2 + y^2 \leq 1 \}.
\]
Let \( f : S^2 \rightarrow S^2 \) be a (continuous) map of degree \( k \). Let \( X \) be the pushout of the diagram
\[
\begin{array}{ccc}
S^2 & \xrightarrow{\text{cof}} & Q \\
\downarrow & & \downarrow \\
\subset & & D^3.
\end{array}
\]
Calculate the homology of \( X \).