(a) Let us rewrite the scheme as

$$\frac{U_i^{n+2} - U_i^n}{2\Delta t} - \frac{U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}}{h^2} = 0$$

Then the truncation error of this scheme is given by (here, $x_i = ih$ and $t_n = n\Delta t$),

$$\begin{aligned}\tau_i^n &= \frac{u(x_i, t_{n+2}) - u(x_i, t_n)}{2\Delta t} - \frac{u(x_{i-1}, t_{n+1}) - 2u(x_i, t_{n+1}) + u(x_{i+1}, t_{n+1})}{h^2} \\ &= u_t(x_i, t_{n+1}) + O(\Delta t) - u_{xx}(x_i, t_{n+1}) + O(h^2) \\ &= O(\Delta t + h^2)\end{aligned}$$

(b) In matrix form, the scheme can be written as,

$$U^{n+2} = 2\Delta t A U^{n+1} + U^n$$

where A is the central difference discretization of the ∂_x^2 operator, with -2's along the diagonal and 1's along the first super- and sub-diagonal. We can then write down the recurrence relation between the global errors:

$$E^{n+2} = 2\Delta t A E^{n+1} + E^n - 2\Delta t \tau^n$$

Since A is real and symmetric, it has a full spectral decomposition, *i.e.* there exists a orthonormal eigenbasis of A. We can therefore write

$$E_{(k)}^{n+2} = 2\Delta t \lambda_k E_{(k)}^{n+1} + E_{(k)}^n - 2\Delta t \tau_{(k)}^n$$

where λ_k is the k-th eigenvalue of A, and $v_{(k)}$ denotes component of vector v along the corresponding eigenvector. We see that the truncation error $\tau_{(k)}^n$ at timestep n would not grow in the successive timesteps if roots of the characteristic equation

$$\rho^2 = 2\Delta t \lambda_k \rho + 1$$

given by

$$\rho = \Delta t \lambda_k \pm \sqrt{(\Delta t \lambda_k)^2 + 1}$$

satisfies $|\rho| \leq 1$ for each eigevalue λ_k. Now eigenvalues of A are given by,

$$\lambda_k = \frac{2}{h^2}(\cos \frac{k\pi h}{N+1} - 1), \quad k = 1, \ldots, N$$

Thus $\lambda_k < 0$ and it is therefore enough to choose Δt such that

$$-1 \leq \Delta t \lambda_k - \sqrt{(\Delta t \lambda_k)^2 + 1} \leq \Delta t \lambda_k + \sqrt{(\Delta t \lambda_k)^2 + 1} \leq 1$$

The first inequality implies

$$\sqrt{(\Delta t \lambda_k)^2 + 1} \leq 1 + \Delta t \lambda_k \implies (\Delta t \lambda_k)^2 + 1 \leq (1 + \Delta t \lambda_k)^2 \implies \lambda_k \geq 0,$$

a contradiction; thus the method is never stable.