1. Consider the two-point boundary value problem,

$$y'' + ay' + by = c, \quad 0 \leq x \leq 1$$
$$y(0) = \alpha, y(1) = \beta.$$

a) Write the difference equation at an interior point x_i using the forward difference for the first-order and central difference for the second-order derivatives respectively. Use this to derive a matrix equation of the form $Ay = f$.

b) Use central difference scheme for the first order derivative and rewrite the matrix equation.

2 a) Show that $u^p(x) = \sin(p\pi x), \quad p = 1, 2, 3, \ldots$ are eigenfunctions of the second-order differential operator $\frac{\partial^2}{\partial x^2}$ on $[0, 1]$ with homogeneous Dirichlet boundary conditions.

b) Show that for each $p = 1, 2, 3, \ldots$, the vectors $u_j^p = \sin(p\pi jh), j = 1, 2, \ldots, m$ with $h = 1/m$ are eigenvectors of the discrete second-order differential operator A defined by

$$(Au)_j = \frac{1}{h^2}\left(u_{j-1}^p - 2u_j^p + u_{j+1}^p\right).$$

3. Consider the implicit 2-step method $\alpha y_n + \beta y_{n-1} + \gamma y_{n-2} = hf(y_n)$ for the problem $y' = f(y),\ y(0) = y_0$.

a) Find the values of α, β, γ which yield a 2nd order method.

b) Show that the method is stable.

c) Show that the negative real axis is contained in the region of absolute stability.

4. Consider the following method for solving the heat equation $u_t = u_{xx}$:

$$U_i^{n+2} = U_i^n + \frac{2\Delta t}{h^2}(U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}).$$

a) Determine the order of accuracy of this method (in both space and time).

b) Suppose we take $\Delta t = \alpha h^2$ for some fixed $\alpha > 0$ and refine the grid. For what values of α (if any) will this method be Lax-Richtmyer stable and hence convergent?

5. Derive the modified equation for the centered difference scheme

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2h}(u_{j+1}^n - u_{j-1}^n)$$

for the approximation of the advection equation $u_t + au_x = 0$.