Applied Functional Analysis Qualifying exam

Assigned January 9, 2022

Problem 1

Let H_1 and H_2 be Hilbert spaces and $A : H_1 \mapsto H_2$ a compact operator. Prove that adjoint $A^\star$ of A is compact.

Solution

As a consequence of the Riesz representation theorem, A is guaranteed to have a unique adjoint $A^\star : H_2 \mapsto H_1$ and moreover, $A^\star$ is a bounded linear operator.

Take any bounded sequence $(y_n)_{n\geq 1}$ in H_2, and let M be an upper bound

$$\|y_n\|_{H_2} \leq M, \quad \forall\, n \geq 1. \tag{1}$$

Since $A^\star$ is bounded, we can get the following bounded sequence in H_1,

$$(x_n)_{n\geq 1}, \quad x_n := A^\star y_n, \quad \forall\, n \geq 1. \tag{2}$$

Now because A is compact, there exists a subsequence of the bounded sequence defined in (2), call this subsequence $(x_{\sigma(n)})_{n\geq 1}$, where σ denotes the subsequence selection function, such that $(Ax_{\sigma(n)})_{n\geq 1}$ converges in H_2. We can now prove that the sequence $(A^\star y_{\sigma(n)})_{n\geq 1}$ is Cauchy in H_1:

Indeed, we have for all $n, m \geq 1$,

$$\begin{aligned}
\|A^\star y_{\sigma(n)} - A^\star y_{\sigma(m)}\|_{H_1}^2 &= \langle A^\star(y_{\sigma(n)} - y_{\sigma(m)}), A^\star y_{\sigma(n)} - A^\star y_{\sigma(m)}\rangle_{H_1} \\
&= \left|\langle A^\star(y_{\sigma(n)} - y_{\sigma(m)}), x_{\sigma(n)} - x_{\sigma(m)}\rangle_{H_1}\right| \\
&= \left|\langle y_{\sigma(n)} - y_{\sigma(m)}, Ax_{\sigma(n)} - Ax_{\sigma(m)}\rangle_{H_2}\right| \\
&\leq \|y_{\sigma(n)} - y_{\sigma(m)}\|_{H_2}\|Ax_{\sigma(n)} - Ax_{\sigma(m)}\|_{H_2},
\end{aligned}$$

where the first equality is by the definition of the norm, the second equality is because the quantity is positive and by definition (2), the third equality is by the definition of the adjoint and the last line is by the Cauchy-Schwartz inequality. Next, we use (1) to get

$$\|A^\star y_{\sigma(n)} - A^\star y_{\sigma(m)}\|_{H_1}^2 \leq 2M\|Ax_{\sigma(n)} - Ax_{\sigma(m)}\|_{H_2},$$

and conclude from the fact that $(Ax_{\sigma(n)})_{n\geq 1}$ is convergent and therefore Cauchy, that the sequence $(A^\star y_{\sigma(n)})_{n\geq 1}$ is Cauchy and thus convergent in H_1 (because H_1 is complete).

We have now shown: For any bounded sequence $(y_n)_{n\geq 1}$ in H_2, there is a subsequence $(y_{\sigma(n)})_{n\geq 1}$ such that $(A^\star y_{\sigma(n)})_{n\geq 1}$ converges in H_1. This proves that $A^\star$ is compact.

Problem 2

Let a, b be two real numbers satisfying $a < b$ and consider the space $L^2([a, b])$, with the usual inner product

$$\langle f, g\rangle = \int_a^b \overline{f(x)}g(x)dx, \qquad \forall\, f, g \in L^2([a, b]),$$

where the bar denotes complex conjugate. Introduce also the space $H := L^2([a, b] \times [a, b])$ and denote its inner product by

$$\langle F, G\rangle_H = \int_a^b \int_a^b \overline{F(x, y)}G(x, y)dxdy, \qquad \forall\, F, G \in H.$$

1. Let $(\varphi_n(x))_{n\geq 1}$ and $(\tilde{\varphi}_n(x))_{n\geq 1}$ be two orthonormal bases of $L^2([a,b])$. Prove that $(\varphi_n(x)\tilde{\varphi}_m(y))_{n,m\geq 1}$ is an orthonormal basis of H.

2. Consider the compact linear integral operator $A : L^2([a,b]) \mapsto L^2([a,b])$ defined by

$$Af(x) := \int_a^b e^{-(x-y)^2} f(y)dy.$$

Denote its eigenvalues by λ_n and its eigenfunctions by $u_n(x)$, for $n \geq 1$. Prove that the kernel of A satisfies

$$e^{-(x-y)^2} = \sum_{n=1}^{\infty} \lambda_n u_n(x)\overline{u_n(y)},$$

where the series converges in the H norm. Prove also that

$$\int_a^b \int_a^b e^{-2(x-y)^2} dxdy = \sum_{n=1}^{\infty} \lambda_n^2.$$

Solution

Question 1: Let us introduce the notation:

$$\Phi_{n,m}(x,y) := \varphi_n(x)\tilde{\varphi}_m(y), \qquad \forall\, x,y \in [a,b],\ \forall\, n,m \geq 1. \tag{3}$$

Note that

$$\mathcal{F} := \{\Phi_{n,m}(x,y),\ n,m \geq 1\} \tag{4}$$

is a countable, orthonormal family in H, because

$$\langle \Phi_{n,m}, \Phi_{n',m'} \rangle_H = \int_a^b \int_a^b \overline{\Phi_{n,m}(x,y)} \Phi_{n',m'}(x,y)dxdy = \langle \varphi_n, \varphi_{n'} \rangle \langle \tilde{\varphi}_m, \tilde{\varphi}_{m'} \rangle = \delta_{n,n'}\delta_{m,m'}.$$

Here we used Fubini's theorem to write the double integral as an iterated integral (it is easy to show that the theorem applies). We also used definition (3) and the assumed orthonormality of the bases of $L^2([a,b])$. To show that the family (4) is a basis of H, we now prove that if $\Psi \in H$ satisfies

$$\langle \Psi, \Phi_{n,m} \rangle_H = 0, \qquad \forall\, n,m \geq 1, \tag{5}$$

then $\Psi = 0$:

Let us introduce the functions

$$\eta_n(y) := \int_a^b \Psi(x,y)\varphi_n(x)dx, \qquad \forall, n \geq 1. \tag{6}$$

Using the definition (3) of $\Phi_{n,m}$ and Fubini's theorem, we conclude from (5) that each of the η_n satisfy

$$\langle \eta_n, \tilde{\varphi}_m \rangle = 0, \qquad \forall\, m \geq 1. \tag{7}$$

Since we know that $\left(\tilde{\varphi}_m(y)\right)_{m\geq 1}$ is a basis of $L^2([a,b])$, we must have $\eta_n(y) = 0$ for all $n \geq 1$ and a.e. in y. But then, we conclude from (6) and the fact that $\left(\varphi_n(x)\right)_{n\geq 1}$ is a basis of $L^2([a,b])$ that $\Psi(x,y) = 0$ a.e.

Question 2: The result follows from the fact that A is self-adjoint and compact. First, since A is compact, it is also bounded and is guaranteed to have a unique adjoint operator $A^\star$. That $A = A^\star$ follows from

$$\langle f, Ag \rangle = \int_a^b \overline{f(x)} \int_a^b e^{-(x-y)^2} g(y)dydx = \int_a^b \overline{\left[\int_a^b e^{-(x-y)^2} f(x)dx\right]} g(y)dy = \langle Af, g \rangle.$$

The spectral theorem for self-adjoint compact operators gives that the eigenvectors $(u_n(x))_{n\geq 1}$ of A form an orthonormal basis of $L^2([a,b])$. That the spectrum of A consists of 0 and real valued eigenvalues also follows from the spectral theorem.

From question 1, with $\varphi_n(x) = u_n(x)$ and with $\tilde{\varphi}_n(y) = \overline{u_n(y)}$ we conclude that the family

$$\mathcal{F} := \{\Phi_{n,m}(x,y) := u_n(x)\overline{u_n(y)},\ n,m \geq 1\}$$

is an orthonormal basis of $H = L^2([a,b]\times[a,b])$. Since the kernel of A, call it

$$K(x,y) := e^{-(x-y)^2}$$

is a function in H, it can be expanded in this basis

$$K(x,y) = e^{-(x-y)^2} = \sum_{n,m=1}^{\infty} c_{n,m}\Phi_{n,m}(x,y), \tag{8}$$

where the series converges in the norm of H and where

$$\begin{aligned} c_{n,m} &= \int_a^b\int_a^b \overline{\Phi_{n,m}(x,y)}e^{-(x-y)^2}dxdy \\ &= \int_a^b \overline{u_n(x)}\left[\int_a^b e^{-(x-y)^2}u_m(y)dy\right]dx \\ &= \lambda_m\int_a^b \overline{u_n(x)}u_m(x)dx = \lambda_n\delta_{n,m}. \end{aligned}$$

Substituting this in (8) we get the desired expansion of the kernel.

The last result is just Parseval's relation, because

$$\|K\|_H^2 = \int_a^b\int_a^b K^2(x,y)dxdy = \int_a^b\int_a^b e^{-2(x-y)^2}dxdy = \sum_{n,m=1}^{\infty}|c_{n,m}|^2 = \sum_{n=1}^{\infty}|\lambda_n|^2.$$

Problem 3

Let H be a Hilbert space over the complex field $\mathbb{C}$.

1. Consider a sequence $(x_n)_{n\geq 1}$ in H that is weakly Cauchy. This means that for any linear bounded functional $\varphi \in H^\star$, the sequence $\big(\varphi(x_n)\big)_{n\geq 1}$ is Cauchy in $\mathbb{C}$. Define the sequence of linear maps

   $$F_n : H^\star \mapsto \mathbb{C}, \quad F_n(\varphi) := \varphi(x_n), \qquad \forall\,\varphi \in H^\star, \ \ \forall\, n \geq 1$$

   and use it to prove that $(x_n)_{n\geq 1}$ is a bounded sequence.

2. Let S be a set in H with the following property: Every non-empty subset of S has a weak Cauchy sequence. Prove that S is bounded.

Solution

Question 1: We know from the definition of F_n and the fact that $(x_n)_{n\geq 1}$ is weakly Cauchy that $\big(F_n(\varphi)\big)_{n\geq 1}$ is Cauchy in $\mathbb{C}$ and thus convergent, for any $\varphi \in H^\star$. This point-wise convergence of the sequence $(F_n)_{n\geq 1}$ and the Banach-Steinhaus theorem give the uniform boundedness result

$$M := \sup_{n\geq 1}\|F_n\| < \infty. \tag{9}$$

Now we prove that $\|F_n\| = \|x_n\|_H$, for all $n \geq 1$. Indeed, for any $n \geq 1$ we have

$$|F_n(\varphi)| = |\varphi(x_n)| \leq \|\varphi\| \|x_n\|_H, \qquad \forall\, \varphi \in H^\star, \tag{10}$$

which implies

$$\|F_n\| = \sup_{\varphi \neq 0, \varphi \in H^\star} \frac{|F_n(\varphi)|}{\|\varphi\|} \leq \|x_n\|_H. \tag{11}$$

Moreover, for the linear functional $\varphi = \varphi_{x_n} = \langle x_n, \cdot \rangle$ given in the Riesz representation theorem we have

$$\big|F(\varphi_{x_n})\big| = \big|\varphi_{x_n}(x_n)\big| = \big|\langle x_n, x_n \rangle\big| = \|x_n\|_H^2 \tag{12}$$

and in addition, $\|\varphi_{x_n}\| = \|x_n\|_H$. Therefore,

$$\frac{\big|F(\varphi_{x_n})\big|}{|\varphi_{x_n}\|} = \|x_n\|_H, \tag{13}$$

and using (11) we get $\|F_n\| = \|x_n\|_H$. The proof follows from (9).

Question 2: We argue by contradiction. Suppose that S is not bounded. Then, we can find a sequence $(x_n)_{n \geq 1}$ in S with the property

$$\|x_n\|_H > n, \qquad \forall\, n \geq 1. \tag{14}$$

Consider the set

$$X := \{x_n,\ n \geq 1\}, \tag{15}$$

which is obviously a non-empty subset of S. By the assumption, there is a weak Cauchy sequence in X. This must be a subsequence $(x_{\sigma(n)})_{n \geq 1}$ of $(x_n)_{n \geq 1}$. But by question 1 this weak Cauchy subsequence must be bounded, which contradicts that $\|x_{\sigma(n)}\| > \sigma(n) \to \infty$ as $n \to \infty$.

We have reached a contradiction, so S must be bounded.

Problem 4

Let δ_y denote the Dirac delta located at $y \in \mathbb{R}$, and let $\delta_y^{(n)}$ denote its n^{th} distributional derivative.

1. Does $\sum_{n=1}^{\infty} \delta_{1/n}^{(n)}$ define a distribution in $\mathscr{D}'(\mathbb{R})$? Prove or disprove.
2. Does $\sum_{n=1}^{\infty} \delta_{1/n}^{(n)}$ define a distribution in $\mathscr{D}'((0,\infty))$? Prove or disprove.
3. Let $f(x)$ be a $C^\infty(\mathbb{R})$ function with $f(0) = 0$ and let $F \in \mathscr{D}'(\mathbb{R})$ be a distribution with support $\{0\}$. Is $f(x)F$ the zero distribution? Prove or disprove.

Solution

Question 1: No, it does not. For instance, there exists a test function $\phi \in \mathscr{D}(\mathbb{R})$ (infinitely differentiable, with compact support) that agrees exactly with e^{-x} for $-1 < x < 1$. Then obviously $\phi^{(n)}(x) = (-1)^n \mathrm{e}^{-x}$ for $-1 < x < 1$. Therefore $(-1)^n \phi^{(n)}(n^{-1}) = \mathrm{e}^{-n^{-1}}$ which is positive and greater than or equal to $\mathrm{e}^{-1} > 0$ for all $n = 1, 2, 3, \ldots$. So, applying the purported distribution to ϕ gives

$$\left(\sum_{n=1}^{\infty} \delta_{1/n}^{(n)} \right) [\phi] = \sum_{n=1}^{\infty} (-1)^n \phi^{(n)}(n^{-1}) \geq \sum_{n=1}^{\infty} \mathrm{e}^{-1}$$

which obviously diverges. So the action of the purported distribution is not even defined on every test function in $\mathscr{D}(\mathbb{R})$.

Question 2: Yes, it does. Firstly, it obviously defines a linear functional on $\mathscr{D}((0,\infty))$. Indeed, every function $\phi \in \mathscr{D}((0,\infty))$ is a C^∞ function with compact support bounded away from zero. Hence

$$\left(\sum_{n=1}^{\infty} \delta_{1/n}^{(n)}\right)[\phi] = \sum_{n=1}^{\infty}(-1)^n \phi^{(n)}(n^{-1}) = \sum_{n=1}^{N(\phi)} \phi^{(n)}(n^{-1})$$

where $N(\phi)$ is the largest positive integer n for which n^{-1} lies in the support of ϕ. This number is necessarily finite by compact support in $(0,\infty)$, so it is a finite sum, which is clearly finite. Linearity is obvious. To prove continuity, suppose that $\phi_k \in \mathscr{D}((0,\infty))$ for $k = 0,1,2,3,\dots$, and that $\phi_k \to 0$ in $\mathscr{D}((0,\infty))$. Then according to the topology of $\mathscr{D}((0,\infty))$, there exists a fixed compact set $K \subset (0,\infty)$ containing the supports of all ϕ_k, and for every $\alpha = 0,1,2,3,\dots$, $\sup_{x>0}|\phi_k^{(\alpha)}(x)| \to 0$ as $k \to \infty$. Then

$$\left(\sum_{n=1}^{\infty} \delta_{1/n}^{(n)}\right)[\phi_k] = \sum_{n=1}^{\infty}(-1)^n \phi_k^{(n)}(n^{-1}) = \sum_{n=1}^{N}(-1)^n \phi_k^{(n)}(n^{-1})$$

where N is the largest positive integer n for which n^{-1} lies in K. This number is necessarily finite because $K \subset (0,\infty)$ is compact. Therefore,

$$\left|\left(\sum_{n=1}^{\infty} \delta_{1/n}^{(n)}\right)[\phi_k]\right| \le \sum_{n=1}^{N} \sup_{x>0}|\phi_k^{(n)}(x)| \to 0, \quad k \to \infty,$$

which establishes the continuity of the functional.

Question 3: Not necessarily. For instance, let $f(x) := x$ and let $F = \delta'$. Then $f(x)F$ acts on a test function $\phi \in \mathscr{D}(\mathbb{R})$ by

$$(f(x)F)[\phi] = F[f\phi] = -(f\phi)'(0) = -f(0)\phi'(0) - f'(0)\phi(0) = -f'(0)\phi(0) = -\phi(0)$$

so $f(x)F = -\delta_0 \neq 0$ in this case.

Problem 5

Suppose that A is a compact self-adjoint operator on a Hilbert space H, and suppose that the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of A are all nonzero and satisfy $|\lambda_n| \le Cn^{-1}$ for some $C > 0$. Let $\{e_n\}_{n=1}^{\infty}$ denote the corresponding orthonormal eigenvectors.

1. Show that $f := \sum_{n=1}^{\infty} n^{-1/2}\lambda_n e_n \in H$.
2. Is $f \in \mathrm{Ran}(A)$? Prove or disprove.

Solution

Question 1: The generalized Fourier coefficients of f are $c_n := n^{-1/2}\lambda_n$. Since $|c_n| \le Cn^{-3/2}$, we have $|c_n|^2 \le C^2 n^{-3}$ which is summable, so $\{c_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N})$. Therefore $f \in H$.

Question 2: No, $f \notin \mathrm{Ran}(A)$. Indeed, if $f \in \mathrm{Ran}(A)$, then there exists $u \in H$ with $Au = f$. Expanding u in the orthonormal basis of eigenvectors of A gives

$$u = \sum_{n=1}^{\infty} u_n e_n \implies Au = \sum_{n=1}^{\infty} \lambda_n u_n e_n$$

so equating $Au = f$ shows that $\lambda_n u_n = n^{-1/2}\lambda_n$. Since no eigenvalues are zero, $u_n = n^{-1/2}$. But then $|u_n|^2 = n^{-1}$ which is not summable, so we arrive at a contradiction with $u \in H$.