1. Consider the initial-value problem for $y(t)$:

$$
y' = \frac{t}{y}
$$

$$
y(0) = 2
$$

$$
0 \leq t \leq 5
$$

(a) Solve the initial-value problem exactly.

(b) Is the problem well-posed? Justify.

(c) Approximate the solution on $[0, 2]$ using the Euler method with step size $\Delta t = 1$. Put $w_0 = y(0)$ and approximate $w_i \approx y(i)$.

(d) Sketch $f(t, y) = \frac{t}{y}$ on the $t$-$y$ plane by putting an arrow at $(t, y)$ with $\Delta t = 1$ and $\Delta y = f(t, y)$, at representative and useful points $(t, y)$. Sketch the exact solution, asymptotes, and the numeric solution to the IVP on $[0, 2]$. Indicate geometrically why the problem is or is not well-posed.
2. The modified Euler method for the initial-value problem \( y'(t) = f(t, y) \); \( y(a) = \alpha \) on \( t \in [a, b] \) is as follows:
\[
\tilde{w} = w_i + \frac{h}{2}f(t_i, w_i) \\
w_{i+1} = w_i + hf(t_i + \frac{h}{2}, \tilde{w})
\]
Derive the local truncation error, for “reasonable” functions \( f \). (Answer in the form \( O(h^k) \).) Give appropriate conditions on \( f \) to achieve that order, and explain.

3. Consider the boundary value problem
\[
-u'' + \pi^2 u = 2\pi^2 \sin(\pi x) \\
u(0) = u(1) = 0.
\]
(a) Set up a finite difference approximation with \( h = \Delta x = \frac{1}{4} \) as a system of equations in unknowns \( w_1 \approx u(h), w_2 \approx u(2h), w_3 \approx u(3h) \). Use the central form of the second difference. (You can include “unknowns” \( w_0 \) and \( w_4 \) if that makes it cleaner to set up the system.)
(b) Set up a Jacobi iteration to solve this system. (No need to solve by hand.)
(c) Set up a Gauss-Seidel iteration.
(d) Comment on the convergence and efficiency of the iterative schemes.
(e) What properties of the iterative schemes or their convergence would be different if the term \( \pi^2 u \) were instead \(-32u\)? What if instead of \( \pi^2 u \) it were \(-\pi^2 u\)?
4. Consider the initial boundary value problem

\[ u_t = \frac{1}{16} u_{xx} \]

\[ u(0, t) = u(1, t) = 0 \]

\[ u(x, 0) = 2 \sin(2\pi x) \]

(a) Sketch the domain on the x-t plane with x horizontal and t vertical.

(b) Set up a continuous-time and central-space discretization

\[ \frac{d\vec{v}}{dt} = -\frac{D}{(\Delta x)^2} \left[ A\vec{v}(t) + \vec{b}(t) \right]. \]

That is, find the scalar \( D \), the matrix \( A \), and the vector \( \vec{b} \). (The vector \( \vec{v}(t) \) approximates the solution \( \vec{u}(t) \) discretized in space at time \( t \).) Use a central difference for the discretization of \( u_{xx} \).

(c) Now discretize your system in time to first order, using \( \vec{w}(n) \) as unknown vectors and \( n \) corresponding to discrete time. Use the forward-time discretization, which result in an overall Forward-Time, Central-Space scheme. Derive the evolution \[ \vec{w}(n+1) = (I - \lambda A)\vec{w}(n) - \lambda \vec{b}(n+1), \]

where the superscript indicates discrete time. Comment on the value of \( \lambda \) and its role in convergence.

(d) Explicitly solve for \[ \begin{bmatrix} w_1^{(n)} \\ w_2^{(n)} \\ w_3^{(n)} \end{bmatrix} \] (omitting \( w_0 = w_4 = 0 \)), using \( \Delta x = \Delta t = \frac{1}{4} \). Write the values of \( \vec{w}^{(n)} \) for \( n = 0, 1, 2 \).

5. Consider the wave equation

\[ u_{tt} = \frac{1}{25} u_{xx} \]

\[ 0 < x < 1 \]

\[ 0 < t < \infty \]

\[ u(0, t) = -\sin(t/5) \]

\[ u(1, t) = \sin(1 - t/5) \]

\[ u(x, 0) = \sin x \]

\[ u_t(x, 0) = -\frac{1}{5} \cos x. \]

We will set up a discretization \( w_j^{(n)} \approx (u(j\Delta x, n\Delta t) \) by

\[ \frac{w_j^{(n+1)} - 2w_j^{(n)} + w_j^{(n-1)}}{(\Delta t)^2} = \frac{1}{25} \frac{w_{j-1}^{(n)} - 2w_j^{(n)} + w_{j+1}^{(n)}}{(\Delta x)^2}. \]

(a) For given \( \Delta x \), give bounds on \( \Delta t \) to make the discretization stable in \( \ell_2 \).

(b) Explain how to compute \( w_j^{(1)} \) as a special case.