QUALIFYING REVIEW EXAM: APPLIED FUNCTIONAL ANALYSIS JANUARY 2021

(1) Suppose that $f : [-1, 1] \to \mathbb{R}$ is continuous. Find directly the solution $u(x)$ of the boundary-value problem

$$-u''(x) = f(x), \quad u(-1) = u(1) = 0,$$

and show that the solution can be written in the form

$$u(x) = \int_{-1}^{1} K(x, y) f(y) \, \mathrm{d}y$$

for some kernel $K(x, y)$. Then prove that the nonlinear boundary-value problem

$$-u''(x) + \mu \sin(u(x)) = f(x), \quad u(-1) = u(1) = 0$$

has a unique solution provided that $\mu \in \mathbb{R}$ is sufficiently small.

(2) Suppose that $\lambda \in \mathbb{C}$ belongs to the residual spectrum of a bounded linear operator L on a Hilbert space $\mathcal{H}$. Determine whether the complex conjugate λ^* is in the point spectrum, continuous spectrum, residual spectrum, or resolvent set of the adjoint L^*.

(3) Consider the set $C[0, 1]$ of continuous functions $[0, 1] \to \mathbb{C}$. Define a "norm" on $C[0, 1]$ by

$$\|f\| := \sup_{x \in [0,1]} x^2 |f(x)|.$$

Does this make $C[0, 1]$ a normed linear space? Is it a Banach space? Prove your answers.

(4) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with nonempty compact support. Is the linear multiplication operator $M_f : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by $M_f u(x) := f(x)u(x)$ compact? Prove or disprove.

(5) For each $\epsilon > 0$ the function $f_\epsilon(x) := (x + \mathrm{i}\epsilon)^{-1}$ defines a regular distribution on $\mathbb{R}$. Does f_ϵ have a distributional limit as $\epsilon \downarrow 0$? If so, prove it, and find the limiting distribution. (You may work in either $\mathcal{D}'(\mathbb{R})$ or with tempered distributions in $\mathcal{S}'(\mathbb{R})$.)