Applied Functional Analysis QR Exam - August 2021

Problem 1 Consider the sequence of integrable functions $(f_n)_{n\geq 1}$, where

$$f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(x) = -\frac{2n^{3/2}x}{\pi(1+nx)^2}, \qquad n = 1, 2, \dots. \tag{1}$$

1. Prove that the sequence $(f_n)_{n\geq 1}$ converges in the sense of distributions to $\delta'(x)$, the derivative of the Dirac delta distribution.
2. Does the sequence $(f_n)_{n\geq 1}$ converge pointwise? Does it converge uniformly?

Solution

1. Each function in the sequence generates a regular distribution i.e., a continuous linear functional on the space $\mathcal{D}$ of test functions, denoted by

$$\langle f_n, \varphi \rangle = \int_{\mathbb{R}} f_n(x)\varphi(x)dx, \quad \forall \varphi \in \mathcal{D}.$$

We must prove that

$$\lim_{n\to\infty} \langle f_n, \varphi \rangle = -\varphi'(0), \qquad \forall \varphi \in \mathcal{D}.$$

We are interested in test functions φ whose support contain the origin. If that is not the case, it is easy to see from the calculations below that the limit is zero. Let φ be a test function with support contained in $[-\alpha, \beta]$, where $\alpha, \beta > 0$. Note that

$$f_n(x) = -\frac{2n^{3/2}x}{\pi(1+nx)^2} = \frac{d}{dx}\frac{\sqrt{n}}{\pi(1+nx^2)},$$

and use integration by parts to get

$$\begin{aligned}
\langle f_n, \varphi \rangle &= \int_{\mathbb{R}} f_n(x)\varphi(x)dx \\
&= \int_{\mathbb{R}} \frac{d}{dx}\left[\frac{\sqrt{n}}{\pi(1+nx^2)}\right]\varphi(x)dx \\
&= -\int_{\mathbb{R}} \frac{\sqrt{n}}{\pi(1+nx^2)}\varphi'(x)dx.
\end{aligned}$$

Then, using that

$$\int_{\mathbb{R}} \frac{\sqrt{n}}{\pi(1+nx^2)}dx = 1,$$

and the support of φ, we get

$$\begin{aligned}
\langle f_n, \varphi \rangle + \varphi'(0) &= \int_{\mathbb{R}} \frac{\sqrt{n}}{\pi(1+nx^2)}\left[\varphi'(0) - \varphi'(x)\right] \\
&= \varphi'(0)\left[\int_{-\infty}^{-\alpha} \frac{\sqrt{n}}{\pi(1+nx^2)}dx + \int_{\beta}^{\infty} \frac{\sqrt{n}}{\pi(1+nx^2)}dx\right] + \int_{-\alpha}^{\beta} \frac{\sqrt{n}}{\pi(1+nx^2)}\left[\varphi'(0) - \varphi'(x)\right]dx.
\end{aligned}$$

Taking absolute values and using the triangle inequality,

$$|\langle f_n, \varphi \rangle + \varphi'(0)| \leq |\varphi'(0)|\left[\int_{-\infty}^{-\alpha} \frac{\sqrt{n}}{\pi(1+nx^2)}dx + \int_{\beta}^{\infty} \frac{\sqrt{n}}{\pi(1+nx^2)}dx\right] + \int_{-\alpha}^{\beta} \frac{\sqrt{n}}{\pi(1+nx^2)}\left|\varphi'(0) - \varphi'(x)\right|dx.$$

The first two integrals can be evaluated explicitly,

$$\int_{-\infty}^{-\alpha} \frac{\sqrt{n}}{\pi(1+nx^2)}dx = \frac{1}{2} - \frac{\arctan(\alpha\sqrt{n})}{\pi},$$
$$\int_{\beta}^{\infty} \frac{\sqrt{n}}{\pi(1+nx^2)}dx = \frac{1}{2} - \frac{\arctan(\beta\sqrt{n})}{\pi},$$

and both tend to 0 as $n \to \infty$. We also have by the mean value theorem that there exists a constant $C > 0$ such that

$$\big|\varphi'(0) - \varphi'(x)\big| \le C|x|, \qquad x \in [-\alpha, \beta].$$

Here we used that $\varphi''(x)$ is continuous (by definition of $\mathcal{D}$) and thus bounded on the compact interval $[-\alpha, \beta]$. Therefore,

$$\begin{aligned}\int_{-\alpha}^{\beta} \frac{\sqrt{n}}{\pi(1+nx^2)}\big|\varphi'(0) - \varphi'(x)\big|dx &\le C \int_{-\alpha}^{\beta} \frac{\sqrt{n}|x|}{\pi(1+nx^2)}dx \\ &= \frac{1}{\sqrt{n}\pi} \int_{-\sqrt{n}\alpha}^{\sqrt{n}\beta} \frac{|t|dt}{1+t^2} \\ &= \frac{\ln(1+n\alpha) + \ln(1+n\beta)}{\sqrt{n}\pi}\end{aligned}$$

which tends to 0 as $n \to \infty$. We now have the result.

2. The sequence $(f_n)_{n\ge1}$ converges pointwise to the function 0. Indeed, $f_n(0) = 0$ by definition and for any $x \ne 0$ we have

$$|f_n(x)| = \frac{2|x|}{\pi(n^{-3/4} + n^{1/4}x)^2} \to 0, \quad \text{as } n \to \infty.$$

However the convergence is not uniform, because if it was, then $(f_n)_{n\ge1}$ would have converged to the zero distribution.

Problem 2

1. Let $g : X \mapsto X$ be a mapping of a Banach space X into itself. Suppose that there exists a closed ball $\overline{B(x_0, R)}$ contained in X, centered at x_o and of radius R, where g satisfies

$$\|g(x) - g(y)\| \le C\|x - y\|, \qquad \forall\, x, y \in \overline{B(x_0, R)}, \tag{2}$$

for a constant $C \in (0, 1)$. Suppose also that

$$\|g(x_o) - x_o\| < (1 - C)R. \tag{3}$$

Prove that the sequence $(x_n)_{n\ge1}$ in X, defined by $x_n = g(x_{n-1})$ for all $n \ge 1$, converges to a point $x \in \overline{B(x_0, R)}$, which is the unique fixed point of $g(x)$ in the closed ball.

2. Use the result above to set up an iteration for finding a root of the polynomial $x^3 - 4x - 1$.

Solution

1. We know by assumption (3) that x_0 and x_1 are in the ball $\overline{B(x_0, R)}$. Let us show that the whole sequence lies in the ball. We have from (2)-(3) and the triangle inequality that

$$\begin{aligned}\|x_2 - x_0\| &= \|g(x_1) - g(x_0) + g(x_0) - x_0\| \le C\|x_1 - x_0\| + \|x_1 - x_0\| \\ &= (1 + C)\|x_1 - x_0\| = \frac{(1 - C^2)}{(1 - C)}\|x_1 - x_0\| < (1 - C^2)R < R,\end{aligned}$$

and thus $x_2 \in \overline{B(x_0,R)}$.

We proceed inductively

Hypothesis: Suppose that

$$\|x_j - x_0\| \le \frac{(1-C^j)}{(1-C)}\|x_1 - x_0\| < (1-C^j)R,$$

for $j = 1, 2, \ldots, n$. This means in particular that $x_j \in \overline{B(x_0,R)}$ for $j = 1, \ldots, n$.

Inductive step: For x_{n+1} we have using the hypotesis and (2) that

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|g(x_n) - g(x_{n-1})\| \le C\|x_n - x_{n-1}\| = C\|g(x_{n-1}) - g(x_{n-2})\| \\ &\le C^2\|x_{n-1} - x_{n-2}\| \ldots \le C^n\|x_1 - x_0\|,\end{aligned}$$

and therefore, by the triangle inequality, the hypothesis and (3) we get

$$\begin{aligned}\|x_{n+1} - x_0\| &\le \|x_{n+1} - x_n\| + \|x_n - x_0\| \le C^n\|x_1 - x_0\| + \frac{(1-C^n)}{(1-C)}\|x_1 - x_0\| \\ &= \frac{(1-C^{n+1})}{(1-C)}\|x_1 - x_0\| < (1-C^{n+1})R.\end{aligned}$$

This proves, by the principle of induction, that $x_n \in \overline{B(x_0,R)}$ for all $n \ge 1$.

We also have that the sequence $(x_n)_{n\ge1}$ is Cauchy, because for all $m > n \ge 1$,

$$\begin{aligned}\|x_m - x_n\| &\le \|x_m - x_{m-1}\| + \ldots \|x_{n+1} - x_n\| \le \big(C^m + \ldots + C^n\big)\|x_1 - x_0\| \\ &= C^n\big(1 + \ldots + C^{m-n}\big)\|x_1 - x_0\| = \frac{C^n(1 - C^{m-n+1})}{1-C}\|x_1 - x_0\| < C^n R.\end{aligned}$$

Since $C^n \to 0$ as $n \to \infty$, for all $\epsilon > 0$, there exists natural number N such that

$$C^n < \frac{\epsilon}{R}, \qquad \forall\, n \ge N,$$

and therefore, by the above,

$$\|x_m - x_n\| < \epsilon, \qquad \forall\, m \ge n \ge N.$$

The sequence is indeed Cauchy and since X is complete, there is $x \in X$ such that $\lim_{n\to\infty} x_n = x$. But since the sequence is in the closed ball $\overline{B(x_0,R)}$, we must have $x \in \overline{B(x_0,R)}$.

Obviously, g is continuous in $\overline{B(x_0,R)}$ by (2), so

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} g(x_n) = g\big(\lim_{n\to\infty} x_n\big) = g(x),$$

so x is a fixed point of the mapping g in $\overline{B(x_0,R)}$. It is trivial to show that there is no other fixed point, because if there were, call it ξ, we would have

$$\|x - \xi\| = \|g(x) - g(\xi)\| \le C\|x - \xi\| \Longrightarrow (1-C)\|x - \xi\| \le 0 \Longrightarrow \|x - \xi\| = 0 \Longrightarrow x = \xi.$$

We found the unique fixed point of the mapping $g(x)$ restricted to $\overline{B(x_0,R)}$.

2. The polynomial $x^3 - 4x - 1$ has three real roots. We proceed as above, with $x_0 = 0$, which is not a root, and $X = [-\frac{3}{2}, \frac{3}{2}]$. Note that $f(x) = x^3 - 4x - 1$ satisfies

$$f\left(\frac{3}{2}\right) < 0 < f\left(-\frac{3}{2}\right)$$

so there is a root in X. Let us define the function

$$g : X \mapsto X, \qquad g(x) = \frac{1}{x^2 - 4}$$

and note that it satisfies

$$|g(x) - g(y)| = \frac{|y^2 - x^2|}{(4 - x^2)(4 - y^2)} \leq \frac{|x| + |y|}{(4 - x^2)(4 - y^2)}|x - y| \leq \frac{2}{9}|x - y|, \qquad \forall x, y \in [-1, 1]. \tag{4}$$

Thus, we can apply the result at part 1, with $x_0 = 0$, the closed ball $\overline{B(x_0, R)} = [-1, 1]$ and $C = 2/9$ to conclude that the iteration

$$x_{n+1} = g(x_n) = \frac{1}{x_n^2 - 4}, \qquad n \geq 1,$$

converges to the fixed point x of g in $[-1, 1]$. That is the root we look for, because

$$x = g(x) = \frac{1}{x^2 - 4} \Longrightarrow (x^2 - 4)x - 1 = x^3 - 4x - 1 = 0.$$

Here's an alternate solution (maybe more obvious). One could write $x^3 - 4x - 1 = 0$ in the form $x = g(x)$ with $g(x) := \frac{1}{4}(x^3 - 1)$. Then we have

$$|g(x) - g(y)| = \tfrac{1}{4}|x^2 + xy + y^2||x - y|.$$

If we take $x_0 = 0$, then on $\overline{B(x_0, R)}$ we have $|g(x) - g(y)| \leq \frac{3}{4}R^2|x - y| = C|x - y|$ with $C := \frac{3}{4}R^2$. So for the contraction condition $C \in (0, 1)$ to hold we assume that $R < 2/\sqrt{3}$. Also, from $x_0 = 0$ we have

$$|g(x_0) - x_0| = \tfrac{1}{4}.$$

Therefore to satisfy the conditions we need to find some $R \in (0, 2/\sqrt{3})$ such that $(1 - C)R = R - \frac{3}{4}R^3 > \frac{1}{4}$. Obviously $R - \frac{3}{4}R^3$ vanishes at the endpoints $R = 0, 2/\sqrt{3}$ and is positive in between. By the quadratic formula, the critical points are $R = \pm 2/3$, so there is exactly one critical point, a local maximizer, in $(0, 2/\sqrt{3})$. At the maximizing point,

$$(1 - C)R|_{R=2/3} = R - \tfrac{3}{4}R^3\big|_{R=2/3} = \left(\tfrac{2}{3}\right)^2 = \tfrac{4}{9} > \tfrac{1}{4}.$$

So if we take the radius of the ball to be $R = 2/3$, then all of the hypotheses hold and the iteration converges to a unique root in the ball.

<u>Problem 3</u> Consider the linear operator $L : \ell^2 \mapsto \ell^2$ defined on the Hilbert space of square summable sequences by

$$Lx = \left(\frac{\xi_j}{\sqrt{j}}\right)_{j\geq 1}, \qquad \forall\, x = (\xi_j)_{j\geq 1} \in \ell^2. \tag{5}$$

Find the point spectrum, the continuous spectrum and the residual spectrum of L.

Solution

We begin by showing that L is a compact operator: Consider the sequence of operators $(L_n)_{n\geq 1}$, defined by

$$L_n : \ell^2 \mapsto \ell^2, \qquad L_n x = \left(\xi_1, \frac{\xi_2}{\sqrt{2}}, \ldots, \frac{\xi_n}{\sqrt{n}}, 0, 0, \ldots\right), \qquad \forall\, x = (\xi_j)_{j\geq 1} \in \ell^2.$$

These operators are bounded, because

$$\|L_n x\|^2 = \sum_{j=1}^{n} \frac{|\xi_j|^2}{j} \leq \sum_{j=1}^{n} |\xi_j|^2 \leq \|x\|^2, \qquad \forall\, x = (\xi_j)_{j\geq 1} \in \ell^2,$$

and since they have finite dimensional range, they are compact. We then have

$$\|(L-L_n)x\|^2 = \sum_{j=n+1}^{\infty} \frac{|\xi_j|^2}{j} \le \frac{1}{n+1} \sum_{j=n+1}^{\infty} |\xi_j|^2 \le \frac{\|x\|^2}{n+1}, \qquad \forall\, x = (\xi_j)_{j\ge 1} \in \ell^2,$$

which implies that in the operator norm we have

$$\|L - L_n\| \le \frac{1}{\sqrt{n+1}}.$$

Therefore, L_n converges to L uniformly (in operator norm) and the limit is compact.

Note that L is also a self-adjoint operator, so its residual spectrum must be empty. From the spectral theorem we know that the spectrum $\sigma(L)$ is the union of the point spectrum $\sigma_p(L)$ and $\{0\}$. Clearly 0 is not an eigenvalue, because $Lx = 0$ is possible only if $x = 0$. Thus, L has the continuous spectrum $\sigma_c(L) = \{0\}$. The point spectrum consists of the eigenvalues $\lambda_j = 1/\sqrt{j}$, for $j = 1, 2, \ldots$, for the eigenvectors x_j having entry 1 in the $j-$th place and zero everywhere else.

Problem 4 Let A be a bounded linear operator defined on a Hilbert space $\mathcal{H}$, and let $B : \mathcal{H} \to \mathcal{H}$,

$B := \mathbb{I} + A^*A$, where A^* denotes the adjoint of A.

1. Show that $\operatorname{ran} B$ is closed.
2. Is B injective? Prove or find a counterexample.
3. Is B onto? Again prove or find a counterexample.
4. Can A fail to have a closed range? Prove or find a counterexample.

Solution

1. To show that $\operatorname{ran} B$ is closed, we will prove that B is bounded away from zero. Note that given $x \in \mathcal{H}$, $\|Bx\|^2 = (Bx, Bx) = (x + A^*Ax, x + A^*Ax) = (x, x) + (x, A^*Ax) + (A^*Ax, x) + (A^*Ax, A^*Ax) = \|x\|^2 + 2(Ax, Ax) + \|A^*Ax\|^2 = \|x\|^2 + 2\|Ax\|^2 + \|A^*Ax\|^2 \ge \|x\|^2$. Therefore $\|Bx\| \ge \|x\|$ proving that B is bounded below. This is equivalent to $\operatorname{ran} B$ being closed and $Bx = 0$ if and only if $x = 0$.
2. The same argument also proves that B is injective. However, here is a direct proof of the latter: suppose that $Bx = By$ for $x, y \in \mathcal{H}$. Then $Bz = 0$ where $z := x - y$. Now using $Bz = z + A^*Az = 0$, we get $\|z\|^2 = (z, z) = (-A^*Az, z) = -(A^*Az, z) = -(Az, Az) = -\|Az\|^2$. Since $\|z\|^2 \ge 0$ and $\|Az\|^2 \ge 0$, we obtain $\|z\|^2 = 0$ which implies that $z = 0$, i.e., $x = y$.
3. B is indeed onto. That is because from the first result, $\operatorname{ran} B = \overline{\operatorname{ran} B}$, and the latter equals $(\ker B^*)^{\perp}$. But $B^* = B$ and B is injective, so $(\ker B^*)^{\perp} = \mathcal{H}$.
4. Yes, it is possible. Consider $\mathcal{H} = L^2(0,1)$ and $A : \mathcal{H} \to \mathcal{H}$ defined by $Af(x) := \int_0^x f(y)\,\mathrm{d}y$, which is obviously linear. Since $L^2(0,1) \subset L^1(0,1)$ (proof: $\int_0^1 |f(y)|\,\mathrm{d}y \le \sqrt{\int_0^1 |f(y)|^2\,\mathrm{d}y} = \|f\|$ by Cauchy-Schwarz) we have

$$\begin{aligned}\|Af\|^2 = \int_0^1 |Af(x)|^2\,\mathrm{d}x &= \int_0^1 \left|\int_0^x f(y)\,\mathrm{d}y\right|^2 \mathrm{d}x \\ &\le \int_0^1 \left(\int_0^x |f(y)|\,\mathrm{d}y\right)^2 \mathrm{d}x \\ &\le \int_0^1 \left(\int_0^1 |f(y)|\,\mathrm{d}y\right)^2 \mathrm{d}x \\ &\le \int_0^1 \|f\|^2\,\mathrm{d}x = \|f\|^2\end{aligned}$$

so A is defined on all of $\mathcal{H}$ and is bounded with norm $\|A\| \leq 1$. However its range is not closed. To see this, first note that every function in the range of A is absolutely continuous (continuous with L^1 derivative) and vanishes in the limit $x \downarrow 0$. Consider the sequence $\{f_n(x) := n\chi_{(0,n^{-1})}(x)\}_{n=1}^{\infty}$ of piecewise constant functions (χ is the characteristic function of the indicated interval). Obviously $f_n \in \mathcal{H}$ for all n. By direct computation,

$$Af_n(x) = \begin{cases} nx, & 0 < x \leq n^{-1} \\ 1, & n^{-1} \leq x < 1. \end{cases}$$

It is easy to see that $Af_n \to 1$ (limit is the constant function $1 \in \mathcal{H}$) in the $L^2(0,1)$ sense. However the constant function 1 does not vanish as $x \downarrow 0$ so it is not in the range of A. So the range of A is not closed.

Problem 5 Determine whether for every $f \in L^2(\mathbb{R})$ there is a unique solution $u \in L^2(\mathbb{R})$ of

$$u(x) + \int_{\mathbb{R}} \mathrm{e}^{-\frac{1}{2}(x-y)^2} u(y)\,\mathrm{d}y = f(x),$$

and prove it. If the answer is in the affirmative, give a formula for $u(x)$ in terms of f.

Solution

We note that the integral operator has a convolution kernel, so we take the Fourier transform on $L^2(\mathbb{R})$:

$$\hat{u}(k) + \widehat{\mathrm{e}^{-\frac{1}{2}x^2}}(k)\hat{u}(k) = \hat{f}(k).$$

But the kernel is a positive Gaussian so its Fourier transform is as well:

$$\widehat{\mathrm{e}^{-\frac{1}{2}x^2}}(k) = C\mathrm{e}^{-hk^2}$$

for some positive constants $C > 0$, $h > 0$ (depending on the normalization of the Fourier transform). Therefore,

$$\hat{u}(k) = \frac{\hat{f}(k)}{1 + C\mathrm{e}^{-hk^2}}.$$

The right-hand side is in $L^2(\mathbb{R})$, because $\hat{f}$ is by Plancherel, and

$$|\hat{u}(k)| \leq |\hat{f}(k)|$$

because $C\mathrm{e}^{-hk^2} > 0$. Therefore, $\hat{u} \in L^2(\mathbb{R})$, and is uniquely determined. By Plancherel again, $u \in L^2(\mathbb{R})$ is also uniquely determined. This proves unique solvability. A formula for $u(x)$ is given by inverse Fourier transform.