Problem 1 Consider the sequence of integrable functions \((f_n)_{n \geq 1}\), where

\[ f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(x) = -\frac{2n^{3/2}x}{\pi(1 + nx)^2}, \quad n = 1, 2, \ldots \]  

1. Prove that the sequence \((f_n)_{n \geq 1}\) converges in the sense of distributions to \(\delta'(x)\), the derivative of the Dirac delta distribution.

2. Does the sequence \((f_n)_{n \geq 1}\) converge pointwise? Does it converge uniformly?

Problem 2

1. Let \(g : X \mapsto X\) be a mapping of a Banach space \(X\) into itself. Suppose that there exists a closed ball \(B(x_0, R)\) contained in \(X\), centered at \(x_0\) and of radius \(R\), where \(g\) satisfies

\[ \|g(x) - g(y)\| \leq C\|x - y\|, \quad \forall x, y \in B(x_0, R), \]  

for a constant \(C \in (0, 1)\). Suppose also that

\[ \|g(x_0) - x_0\| < (1 - C)R. \]  

Prove that the sequence \((x_n)_{n \geq 1}\) in \(X\), defined by \(x_n = g(x_{n-1})\) for all \(n \geq 1\), converges to a point \(x \in B(x_0, R)\), which is the unique fixed point of \(g(x)\) in the closed ball.

2. Use the result above to set up an iteration for finding a root of the polynomial \(x^3 - 4x - 1\).

Problem 3 Consider the linear operator \(L : \ell^2 \mapsto \ell^2\) defined on the Hilbert space of square summable sequences by

\[ Lx = \left( \frac{\xi_j}{\sqrt{j}} \right)_{j \geq 1}, \quad \forall x = (\xi_j)_{j \geq 1} \in \ell^2. \]  

Find the point spectrum, the continuous spectrum and the residual spectrum of \(L\).

Problem 4 Let \(A\) be a bounded linear operator defined on a Hilbert space \(\mathcal{H}\), and let \(B : \mathcal{H} \to \mathcal{H}\),

\[ B := I + A^*A, \]  

where \(A^*\) denotes the adjoint of \(A\).

1. Show that ran \(B\) is closed.

2. Is \(B\) injective? Prove or find a counterexample.

3. Is \(B\) onto? Again prove or find a counterexample.

4. Can \(A\) fail to have a closed range? Prove or find a counterexample.
Problem 5 Determine whether for every $f \in L^2(\mathbb{R})$ there is a unique solution $u \in L^2(\mathbb{R})$ of

$$u(x) + \int_{\mathbb{R}} e^{-\frac{1}{2}(x-y)^2} u(y) \, dy = f(x),$$

and prove it. If the answer is in the affirmative, give a formula for $u(x)$ in terms of $f$. 