(1) Consider the nonlinear integral equation
\[ u(x) = \lambda \int_0^1 \frac{dy}{1 + x + u(y)}, \quad 0 \leq x \leq 1, \]
where \( \lambda \) is a real parameter and \( u(x) \) is to be found. Prove that if \( 0 \leq \lambda < 1 \), then there exists exactly one solution that is continuous on the interval \([0, 1]\) and satisfies \( u(x) \geq 0 \).

(2) Consider the linear operator \( A \) defined on the Banach space \( L^1(-1, 1) \) with dense domain \( C([-1, 1]) \) and action \( Au(x) := u(x^2) \). Such operators arise in the representation theory of nonlinear maps \( f : (-1, 1) \to (-1, 1) \); in this case, \( f(x) := x^2 \).
   (a) Find the kernel \( \ker(A) \) and range \( \text{ran}(A) \).
   (b) Is \( A \) a bounded operator (with respect to the \( L^1(-1, 1) \to L^1(-1, 1) \) norm)? Prove or disprove.
   (c) Does \( A \) have any nonzero eigenvalues? Prove or disprove.

(3) For each \( N \), the set of monomials \( \{x^n\}_{n=0}^N \) is linearly independent in \( H = L^2(-1, 1) \). The Legendre polynomials \( \{p_n(x)\}_{n=0}^\infty \) are the result of applying the Gram-Schmidt process to \( \{x^n\}_{n=0}^\infty \).
   (a) Find \( p_0(x) \), \( p_1(x) \), and \( p_2(x) \).
   (b) Find the element in the closed linear subspace of \( H \) spanned by \( 1 \), \( x \), and \( x^2 \) that minimizes the distance to \( x^3 \).

(4) Recall that the Fourier transform on the Schwartz space \( S(\mathbb{R}) \) is given by
\[ \hat{\varphi}(k) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x)e^{-ikx} \, dx. \]
For each of the following tempered distributions, (i) find its Fourier transform and express its action on a test function \( \varphi \) directly, i.e., not in terms of \( \hat{\varphi} \), and (ii) state whether the resulting distribution is regular or singular, and if regular identify the function represented.
   (a) \( \delta'(x) \) (derivative of the delta distribution).
   (b) \( \text{sgn}(x) \) (regular distribution given by a function)

(5) Consider the bounded Fredholm operator \( A \) defined on the Hilbert space \( \mathcal{H} = L^2(0, 1) \) by
\[ Au(x) := \int_x^1 u(y) \, dy. \]
   (a) Find the adjoint.
   (b) Use part (a) to determine for which functions \( f \in \mathcal{H} \) there is a solution \( u(x) \) of the equation \( Au + u = f \).
   (c) Suppose that \( u_n \rightharpoonup u \) ("\( \rightharpoonup \)" denotes weak convergence). Using only the boundedness of \( A \), prove that \( Au_n \rightharpoonup Au \).